

THE AUTOMORPHISMS OF TWO-GENERATOR ONE-RELATOR GROUPS WITH TORSION

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ABSTRACT. Let G be a group with presentation of the form $G = \langle a, b; R^n \rangle$, $n > 1$. We classify when $\text{Out}(G)$ is infinite and prove that either $\text{Out}(G)$ is virtually cyclic or $G \cong \langle a, b; [a, b]^n \rangle$ and $\text{Out}(G) = \text{Out}(F(a, b))$. We classify $\text{Out}(G)$ when $R \notin F(a, b)'$, and give an algorithm to find $\text{Out}(G)$ in this case. We also apply our ideas to some other two-generator groups.

1. INTRODUCTION

Let $G = \langle a, b; R^n \rangle$ with R not a true power of any element of $F(a, b)$. Such a group contains torsion elements if and only if $n > 1$. The aim of this paper is to investigate $\text{Out}(G)$ when $n > 1$.

Throughout, n denotes an integer greater than 1, $n > 1$, while R is a freely and cyclically reduced, non-trivial word in $F(a, b)$.

We present three main results.

The first result is that, given a group $G = \langle a, b; R^n \rangle$ such that $R \notin F(a, b)'$, there exists an algorithm to determine $\text{Out}(G)$. If we define the maps,

$$\begin{array}{ll} \alpha_k : a \mapsto a^{-1}b^k & \beta_k : a \mapsto ab^k \\ b \mapsto b & b \mapsto b^{-1} \\ \\ \zeta_k : a \mapsto a^{-1}b^k & \delta_k : a \mapsto ab^k \\ b \mapsto b^{-1} & b \mapsto b \end{array}$$

then the algorithm is as follows:

- Find a cyclically reduced word $S \in F(a, b)$ such that a or b appears with exponent sum zero in S and $G \cong \langle a, b; S^n \rangle$.

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- Rewrite S such that it is the generator a which appears with exponent sum zero.
- If $S \in \{b, b^{-1}\}$ then,
 - $\text{Out}(G) \cong D_n \rtimes \text{Aut}(C_n)$ where elements of $\text{Aut}(C_n)$ commute with the flip-generator and act on the rotations as automorphisms in the natural way.
- Else,
 - Calculate $S\delta_0$. Then $\delta_0 \in \text{Aut}(G)$ if and only if $S\delta_0 \equiv S$ or $S\delta_0 \equiv aSa^{-1}$.
 - If $\delta_0 \notin \text{Aut}(G)$ then $\text{Out}(G)$ is finite, and,
 - * Compute the least i such that $ab^i a$ is a subword of a cyclic shift of R . Call this number \min_+ .
 - * Compute the greatest i such that $ab^i a$ is a subword of a cyclic shift of R . Call this number \max_+ .
 - * Compute the least i such that $a^{-1}b^i a^{-1}$ is a subword of a cyclic shift of R . Call this number \min_- .
 - * Compute the greatest i such that $a^{-1}b^i a^{-1}$ is a subword of a cyclic shift of R . Call this number \max_- .
 - * Calculate $S\zeta_0$, then $\zeta_0 \in \text{Aut}(G)$ if and only if $S\zeta_0 =_G 1$.
 - * Calculate $S\alpha_k$ for all k such that $k = -(\max_+ + \min_+)$ or
 $\min_- - \min_+ \leq k \leq \max_- - \max_+$
then $\alpha_k \in \text{Aut}(G)$ if and only if $S\alpha_k =_G 1$.
 - * Calculate $S\beta_k$ for all k such that $k = \max_+ + \min_+$ or
 $\max_+ - \max_- \leq k \leq \min_+ - \min_-$
then $\beta_k \in \text{Aut}(G)$ if and only if $S\beta_k =_G 1$.
 - * If there exists some k in the above ranges such that $\alpha_k, \beta_{-k}, \zeta_0 \in \text{Aut}(G)$ then

$$\text{Out}(G) \cong C_2 \times C_2$$
 - * Else if $\zeta_0 \in \text{Aut}(G)$ then

$$\text{Out}(G) \cong C_2$$
 - * Else if $\alpha_k \in \text{Aut}(G)$ for some k in the above range then

$$\text{Out}(G) \cong C_2$$
 - * Else if $\beta_k \in \text{Aut}(G)$ for some k in the above range then

$$\text{Out}(G) \cong C_2$$

- * Else, $\text{Out}(G)$ is trivial.
- Else, $\text{Out}(G)$ is infinite and,
 - * Calculate $S\alpha_0$. Then $\alpha_0 \in \text{Aut}(G)$ if and only if $S\alpha_0 =_G 1$.
 - * Calculate $S\beta_0$. Then $\beta_0 \in \text{Aut}(G)$ if and only if $S\beta_0 =_G 1$.
 - * Calculate $S\zeta_0$. Then $\zeta_0 \in \text{Aut}(G)$ if and only if $S\zeta_0 =_G 1$.
 - * Then the only possibilities are as follows:
 - $\text{Out}(G) \cong \mathbb{Z}$ if $\delta_0 \in \text{Aut}(G)$ but $\alpha_0, \beta_0, \zeta_0 \notin \text{Aut}(G)$.
 - $\text{Out}(G) \cong \mathbb{Z} \times C_2$ if $\delta_0, \zeta_0 \in \text{Aut}(G)$ but $\alpha_0, \beta_0 \notin \text{Aut}(G)$.
 - $\text{Out}(G) \cong D_\infty$ if $\delta_0, \alpha_0 \in \text{Aut}(G)$ but $\beta_0, \zeta_0 \notin \text{Aut}(G)$.
 - $\text{Out}(G) \cong D_\infty$ if $\delta_0, \beta_0 \in \text{Aut}(G)$ but $\alpha_0, \zeta_0 \notin \text{Aut}(G)$.
 - $\text{Out}(G) \cong D_\infty \times C_2$ if $\delta_0, \alpha_0, \beta_0, \zeta_0 \in \text{Aut}(G)$.

Note that we prove that if $S \notin \{b, b^{-1}\}$ and $\delta_0 \notin \text{Aut}(G)$ then it is not possible for $\alpha_i, \alpha_j \in \text{Aut}(G)$ with $i \neq j$, nor for $\beta_i, \beta_j \in \text{Aut}(G)$ with $i \neq j$, nor for $\alpha_i, \beta_j \in \text{Aut}(G)$ with $i \neq -j$, and nor for $\zeta_i \in \text{Aut}(G)$ with $i \neq 0$ to occur. Therefore, if $\delta_0 \notin \text{Aut}(G)$ then the only possible cases are as we outlined.

The second result is that if G is a two-generator one-relator group with torsion then $\text{Out}(G)$ finite, virtually- \mathbb{Z} , or $G \cong \langle a, b; [a, b]^n \rangle$ and $\text{Out}(G) = \text{Out}(F(a, b))$.

We also provide a partial result on the outer automorphism groups of arbitrary two-generator one-relator groups, $G = \langle a, b; R \rangle$. We prove that if such a group has only finitely many Nielsen Equivalence Classes in the T -system of (a, b) then $\text{Out}(G)$ is residually finite. Moreover, if $R \notin F(a, b)'$ then $\text{Out}(G)$ is virtually cyclic.

More specific results regarding the residual finiteness of the outer automorphism groups of one-relator groups can be found in, for example, [4], where it is proven that $\text{Out}(G)$ is residually finite if G is the fundamental group of a compact orientable surface of genus k , in [1], where it is proven that $\text{Out}(G)$ is residually finite if G is a cyclically pinched one-relator group, in [8], where the authors characterise the groups of the form $G = \langle a, b; (a^{-s}b^m a^s b^l)^n \rangle$ where $n \geq 1$ with residually finite outer automorphism group, and in [9], where it is proven that $\text{Out}(G)$ is residually finite where $G = \langle a, b; R^n \rangle$, $R \in \{a^l b^m, a^{-s} b^l a^s b^m\}$ and $n > 1$. In each of these cases, the authors apply the ideas of [4]. That

is, $\text{Out}(G)$ is residually finite if G is conjugacy separable and G satisfies Grossman's Property A (if $\gamma \in \text{Aut}(G)$ such that $g\gamma$ is conjugate to g for all $g \in G$ then $\gamma \in \text{Inn}(G)$).

In [3] Gilbert, Howie, Metaftsis and Raptis classify $\text{Out}(G)$ for G a one-relator group with non-trivial centre (they explicitly exclude the free abelian group of rank two). If G is such a group then G is necessarily two-generated and torsion-free, so the main class of groups investigated here is closely related to, but disjoint from, these. However, it is interesting to note that they prove that if G is a one-relator group with trivial centre then $\text{Out}(G)$ is one of C_2 , $C_2 \times C_2$, D_∞ or $D_\infty \times C_2$. This is a very similar classification result to the classification result presented here.

A *primitive element* of a free group F is an element which is contained in some basis for F . A *primitive k -tuple* of F is a k -element subset of a basis of F .

Two n -tuples of a group $G = \langle x_1, \dots, x_n; \mathbf{r} \rangle$, $Y = (y_1, \dots, y_n)$ and $Z = (z_1, \dots, z_n)$ say, are *Nielsen equivalent* if there exists some Nielsen transformation of (x_1, \dots, x_n) , ϕ say, such that if $x_i\phi = w_i(x_1, \dots, x_n)$ then

$$(w_1(Y), \dots, w_n(Y)) =_G (z_1, \dots, z_n)$$

where $=_G$ denotes equality in the group. Further, two n -tuples of G , $Y = (y_1, \dots, y_n)$ and $Z = (z_1, \dots, z_n)$ say, lie in the same *T -system* if there exists some automorphism of G , $\psi \in \text{Aut}(G)$, such that $(y_1\psi, \dots, y_n\psi)$ is Nielsen equivalent to (z_1, \dots, z_n) . Note that Nielsen equivalence and 'lying in the same T -system' are both equivalence relations, and that the T -systems of a group partition the Nielsen equivalence classes.

Along the way we prove that if $G = \langle a, b; R^n \rangle$, $n > 1$ and R is not a primitive element of $F(a, b)$ then $\text{Out}(G)$ embeds into $\text{Out}(F(a, b))$ in a canonical way. This is not surprising in view of the fact that, by [13], the automorphisms of G 'are' automorphisms of $F(a, b)$. That is,

Proposition 1.1. (*Pride, 1977*)

Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and assume R is not a primitive element of $F(a, b)$, or R is a primitive element and $n = 2$. Then G has only one Nielsen Equivalence Class. If R is primitive and $n > 2$ then G has $\frac{1}{2}\varphi(n)$ Nielsen Equivalence Classes (where φ is the Euler totient function).

This means that if $\phi : a \mapsto A, b \mapsto B$ is an automorphism of $G = \langle a, b; R^n \rangle$ then we can, without loss of generality, assume that (A, B) is a primitive pair of $F(a, b)$.

We are also aided by the following two results. The first is a spelling theorem, and can be found in [6],

Proposition 1.2. (*Newman-Gurevich Spelling Theorem*)

Let $G = \langle X; R^n \rangle$, $n > 1$, with $W =_G 1$ but W is freely reduced and not the empty word. Then W contains a Gurevich subword for R^n : a subword $S^{n-1}S_0$ where $S = S_0S_1$ is a cyclic shift of R or R^{-1} , and every generator which appears in R appears in S_0 . If, further, W is cyclically reduced, then either W is a cyclic shift of R^n or R^{-n} , or some cyclic shift of W contains two disjoint subwords, each of which is a Gurevich subword for R^n .

The second result tells us what the Nielsen transformations of the free group on two generators look like, and is a variant of Corollary N4 (p169) of [12],

Proposition 1.3. (*Nielsen, 1924*)

Taking $\phi : a \mapsto A, b \mapsto B$ to be an arbitrary Nielsen transformation of $F(a, b)$, then the map,

$$\xi : \text{Aut}(F(a, b)) \rightarrow GL(2, \mathbb{Z})$$

$$\phi \mapsto \begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ \sigma_a(B) & \sigma_b(B) \end{pmatrix}$$

is an epimorphism, and $\text{Ker}(\xi) = \text{Inn}(F(a, b))$.

Throughout, well-known results on one-relator groups are assumed. These results can be found in Section 4.4 of [12], and Chapters II.5 and IV.5 of [11].

Suppose $G = \langle X; \mathbf{r} \rangle$, $\mathbf{r} \subseteq F(X)$, and let U, V, W be words in $F(X)$, the free group on X . Then $U \equiv V$ will mean that U and V are the same word. If U and V define the same element of G then it will be said that U is equal to V in G , written $U =_G V$, or simply $U = V$ if the group G is understood. For a generator $c \in X^{\pm 1}$ of G , an *exponent of c in W* is an integer e such that $U \equiv Vc^eW$ where neither the last symbol of V nor the first symbol of W are c or c^{-1} . The exponent sum of c in U , denoted $\sigma_c(U)$, is the sum of the exponents of c in U . If U is a freely reduced word then the sum of the absolute values of the exponents of all the generators of G in U is the length of U , and is denoted $|U|$. A word V will be said to be *more than half* of U if V is a subword of U and $|V| > \frac{1}{2}|U|$. A word U will be said to be a *true power* or a *proper power* if there exists a word V and some $n > 1$ such that $U \equiv V^n$.

We will take U^V to mean conjugacy by V . That is, $V^{-1}UV$.

Throughout the paper ϵ (and variations such as ϵ' and ϵ_0) will denote an integer with absolute value 1.

2. NIELSEN EQUIVALENCE OF GENERATING TUPLES

Underlying much of this paper are the notions of Nielsen equivalence of generating tuples and of tame automorphisms, so it is pertinent to go into some detail on them here.

Recall that two n -tuples of a group $G = \langle x_1, \dots, x_n; \mathbf{r} \rangle$, $Y = (y_1, \dots, y_n)$ and $Z = (z_1, \dots, z_n)$ say, are *Nielsen equivalent* if there exists some Nielsen transformation of (x_1, \dots, x_n) , ϕ say, such that if $x_i\phi = w_i(x_1, \dots, x_n)$ then $(w_1(Y), \dots, w_n(Y)) =_G (z_1, \dots, z_n)$.

Let $G = \langle x_1, \dots, x_n; \mathbf{r} \rangle$, then the set

$$L_{(x_1, \dots, x_n)} := \{\phi : \phi \in \text{Aut}(G), (x_1\phi, \dots, x_n\phi) \text{ is Nielsen equivalent to } (x_1, \dots, x_n)\}$$

is the group of *tame automorphisms* of G corresponding to the generating tuple (x_1, \dots, x_n) . Such automorphisms are called *free automorphisms* in [12] and are sometimes also called *lifting automorphisms*.

By Proposition 1.1, if $G = \langle a, b; R^n \rangle$ with $n > 1$ and R non-primitive then $L_{(a,b)} = \text{Aut}(G)$.

Lemma 2.1. *Let G be an arbitrary, finitely generated group defined by the presentation $\langle x_1, \dots, x_n; \mathbf{r} \rangle$. Then, for every Nielsen equivalence class \mathcal{C} of G in the T -system of (x_1, \dots, x_n) there exists some $\psi_c \in \text{Aut}(G)$ with $(x_1\psi_c, \dots, x_n\psi_c) \in \mathcal{C}$ such that if $\tau \in \text{Aut}(G)$ and $(x_1\tau, \dots, x_n\tau) \in \mathcal{C}$ then there exists some $\phi \in L_{(x_1, \dots, x_n)}$ and $\tau = \phi\psi_c$.*

Proof. By the definitions of T -systems and Nielsen equivalence classes, if \mathcal{C} is a Nielsen equivalence class lying in the same T -system of G as (x_1, \dots, x_n) then there exists some automorphism of G which maps (x_1, \dots, x_n) into \mathcal{C} . So, we can pick some $\psi_c \in \text{Aut}(G)$ arbitrarily, with $\psi_c : x_1 \mapsto y_1, \dots, x_n \mapsto y_n$, and $(y_1, \dots, y_n) \in \mathcal{C}$.

Now, let τ be an automorphism of G which takes (x_1, \dots, x_n) to (z_1, \dots, z_n) where $(z_1, \dots, z_n) \in \mathcal{C}$, so $\tau : x_1 \mapsto z_1, \dots, x_n \mapsto z_n$. Then by the definition of Nielsen equivalence class there exists a Nielsen transformation of (x_1, \dots, x_n) , ϕ say, with $x_i\phi = w_i(x_1, \dots, x_n)$, where

$$(y_1, \dots, y_n)\phi = (w_1(y_1, \dots, y_n), \dots, w_n(y_1, \dots, y_n)) =_G (z_1, \dots, z_n).$$

Then,

$$\begin{aligned}
x_i \tau &= z_i \\
&=_G w_i(y_1, \dots, y_n) \\
&= w_i(x_1 \psi_c, \dots, x_n \psi_c) \\
&= w_i(x_1, \dots, x_n) \psi_c \\
&= x_i \phi \psi_c,
\end{aligned}$$

and noting that $\phi = \tau \psi_c^{-1}$ we have that $\phi \in L_{(x_1, \dots, x_n)}$, as required. \square

This lemma tells us that the ψ_c form a set of left coset representatives for $\text{Aut}(G)/L_{(x_1, \dots, x_n)}$. This set is a left transversal, by the definition of $L_{(x_1, \dots, x_n)}$, and so,

$$|\text{Aut}(G) : L_{(x_1, \dots, x_n)}| = \#\text{Nielsen equivalence classes in the } T\text{-system of } (x_1, \dots, x_n).$$

It is necessary that we specify the generating tuple we are dealing with with $L_{(x_1, \dots, x_n)}$, as the set of tame automorphisms can change when we change the generating tuple. For example,

Theorem 2.2. *Let $G = \langle x_1, \dots, x_n; \mathbf{r} \rangle$ be a group which has only finitely many Nielsen equivalence classes in the T -system of (x_1, \dots, x_n) . Then there exists a finite generating set for G , (y_1, \dots, y_m) , such that there is only one Nielsen equivalence class in the T -system of (y_1, \dots, y_m) . Moreover, if \mathbf{r} is finite then there exists a finite set $\mathbf{s} \subset F(y_1, \dots, y_m)$ such that $\langle y_1, \dots, y_m; \mathbf{s} \rangle$ is a finite presentation for G .*

Proof. Let $G = \langle x_1, \dots, x_n; \mathbf{r} \rangle$ be a finitely generated group, and write $X = \{x_1, \dots, x_n\}$. Let $\psi \in \text{Aut}(G)$ where $(x_1 \psi, \dots, x_n \psi)$ is not Nielsen equivalence to (x_1, \dots, x_n) . As Theorem 3.10 (p171) of [12] points out, if $W_i(X) =_G x_i \psi$ then the $2n$ -generated group with presentation,

$$\langle x_1, \dots, x_n, z_1, \dots, z_n; \mathbf{r}, z_1^{-1} W_1(X), \dots, z_n^{-1} W_n(X) \rangle$$

is isomorphic to G , is finitely related if \mathbf{r} is a finite set, and $(x_1 \psi, \dots, x_n \psi, z_1 \psi, \dots, z_n \psi)$ is Nielsen equivalent to $(x_1, \dots, x_n, z_1, \dots, z_n)$. To see this last point, writing $Z = \{z_1, \dots, z_n\}$, note that ψ can now be defines by the mapping $x_i \mapsto z_i$, $z_i \mapsto x_i V_i^{-1}(Z) W_i(Z)$ where $V_i(X)$ is a word on the X such that $V_i(X) \psi = x_i$. This is obviously a Nielsen transformation.

We prove that $|\text{Aut}(G) : L_{(x_1, \dots, x_n)}| > |\text{Aut}(G) : L_{(x_1, \dots, x_n, z_1, \dots, z_n)}|$. This means that after every re-writing of the presentation as above we reduce the index of the Tame automorphisms, and so the number of Nielsen equivalence classes. This suffices, as there are only finitely many Nielsen equivalence classes in the T -system of (x_1, \dots, x_n) .

To do this, we prove that if ϕ is a Nielsen transformation of (x_1, \dots, x_n) which defines an automorphism of G then there is a Nielsen transformation of the $2n$ -tuple $(x_1, \dots, x_n, z_1, \dots, z_n)$, ϕ' say, which defines an automorphism of G and with $x_1\phi' = x_1\phi, \dots, x_n\phi' = x_n\phi$.

So, let ϕ be a Nielsen transformation of (x_1, \dots, x_n) which defines an automorphism of G , and let $V_i(X) =_G x_i\phi$. Then take ϕ' to be the Nielsen transformation $x_i \mapsto V_i(X)$, $z_i \mapsto z_i W_i(X)^{-1} W_i(V_\mu)$ where $W_i(V_\mu)$ is the word W_i over the alphabet $V_\mu(X)$. This Nielsen transformation is an automorphism, and it defines the same automorphism of G as ϕ does, as required. \square

For example, by Proposition 1.1 the group $G = \langle a_1, a_2; a_1^{12} \rangle$ has $\varphi(12) = 4$ Nielsen equivalence classes, but

$$\begin{aligned} G &\cong \langle a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8; \\ &\quad a_1^{12} = 1, \\ &\quad b_1 = a_1^5, b_2 = a_2, \\ &\quad c_1 = a_1^7, c_2 = a_2, c_3 = b_1^7, c_4 = b_2, \\ &\quad d_1 = a_1^{11}, d_2 = a_2, d_3 = b_1^{11}, d_4 = b_2, d_5 = c_1^{11}, d_6 = c_2, d_7 = c_3^{11}, d_8 = c_4 \rangle \end{aligned}$$

and this presentation for G has only one Nielsen equivalence class. Therefore, $L_{(a_1, \dots, d_8)} = \text{Aut}(G)$.

When it is unambiguous which generating tuple we are talking about we shall just write L in place of $L_{(x_1, \dots, x_n)}$.

The reason for the name *lifting automorphism* is because if we define H_X to be the set of Nielsen transformations of $X = (x_1, \dots, x_n)$ which define automorphisms of G then there exists a surjection from H_X to L_X . This means we have Figure 1,

$$\begin{array}{ccc} & \text{Aut}(F(X)) & \\ & \uparrow & \\ H_X & \longrightarrow & L_X \end{array}$$

FIGURE 1

Now, $\text{Inn}(F(X)) \leq H_X$ and elements of $\text{Inn}(F(X))$ induce inner automorphisms of G . Thus, we have Figure 2,

It shall, at various times, be convenient to prove that any two tame automorphisms of G are equal mod $\text{Inn}(G)$ if and only if they are equal

$$\begin{array}{ccc}
& \text{Out}(F(X)) & \\
& \uparrow & \\
\frac{H_X}{\text{Inn}(F(X))} & \xrightarrow{\theta} \twoheadrightarrow & \frac{L_X}{\text{Inn}(G)}
\end{array}$$

FIGURE 2

mod $\text{Inn}(F(X))$. That is, we prove that $\text{Ker}(\theta)$ is trivial. So we have Figure 3,

$$\begin{array}{ccc}
& \text{Out}(F(X)) & \\
& \uparrow & \nwarrow \text{dotted} \\
\frac{H_X}{\text{Inn}(F(X))} & \xrightarrow{\theta} \twoheadrightarrow & \frac{L_X}{\text{Inn}(G)}
\end{array}$$

FIGURE 3

This homomorphism θ is ‘canonical’ in the sense that it is pairing tame automorphisms of G with their associated Nielsen Transformations. Therefore, if we prove that θ is an isomorphism we have proven that $L_X/\text{Inn}(G)$ embeds in $\text{Out}(F(X))$ in a canonical way.

Theorem 2.3. *Let $G = \langle x_1, \dots, x_n; \mathbf{r} \rangle$ be an arbitrary group and let L be the subgroup of tame automorphisms of G . Then $\text{Inn}(G) \trianglelefteq L$, and if,*

- (1) *the T -system of the n -tuple (x_1, \dots, x_n) contains only finitely many Nielsen equivalence classes,*
- (2) *$L/\text{Inn}(G)$ is residually finite,*

then $\text{Out}(G)$ is residually finite.

We also have one more condition,

- (3) *For every Nielsen transformation ϕ such that $\phi \in \text{Aut}(G)$,*

$$(x_1^w, \dots, x_n^w) =_G (x_1\phi, \dots, x_n\phi) \Rightarrow (x_1^w, \dots, x_n^w) \equiv (x_1\phi, \dots, x_n\phi),$$

and the discussion above tells us that (3) \Rightarrow (2).

Proof. Clearly $\text{Inn}(G) \trianglelefteq L$ by the definition of L .

By Lemma 2.1 and condition (1), L is of finite index in $\text{Aut}(G)$. We have that

$$\frac{|\text{Aut}(G)/\text{Inn}(G)|}{|L/\text{Inn}(G)|} = \left| \frac{\text{Aut}(G)}{L} \right|.$$

Therefore, $L/\text{Inn}(G)$ has finite index in $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$. However, $L/\text{Inn}(G)$ is residually finite by assumption. As residually finite groups are closed under finite index, $\text{Out}(G)$ is residually finite. \square

Remark 2.4. *It is interesting to note that if $L \trianglelefteq \text{Aut}(G)$ and we replace condition (1) with the condition,*

- A. *There exists $\psi \in \text{Aut}(G)$ such that for every Nielsen equivalence class \mathcal{C} there exists an integer i_c such that*

$$(x_1, \dots, x_n)\psi^{i_c} \in \mathcal{C}.$$

then Theorem 2.3 still holds.

This is because one can view condition (1) as simply ‘ $\text{Aut}(G)/N$ is finite’, and the fact that conditions (1) and (2) combine to give that $\text{Out}(G)$ is residually finite is because finite-by-(residually finite) groups are residually finite. Condition (A), on the other hand, replaces the condition of ‘ $\text{Aut}(G)/N$ is finite’ with ‘ $\text{Aut}(G)/N \cong \mathbb{Z}$ ’. However, \mathbb{Z} -by-(residually finite) groups are residually finite, and so if conditions (A) and (2) both hold then $\text{Out}(G)$ is residually finite.

By Proposition 1.1, two-generator one-relator groups with torsion satisfy condition (1). In what follows, we prove that condition (2) holds for R primitive (Theorem 4.1) and condition (iii) holds for R non-primitive (Lemma 3.4, Theorem 5.1).

3. $\text{Out}(G)$ FOR $R \notin F(a, b)'$ AND R NON-PRIMITIVE

In this section we wish to study the case where $R \notin F(a, b)'$ and R is not primitive. It is easy to verify if a word $R = W(a, b)$ is in $F(a, b)'$; one just checks that $\sigma_a(W) = 0 = \sigma_b(W)$. On the other hand, it is well-known that there exists an algorithm to re-write $G = \langle a, b; R^n \rangle$ as $\langle a, b; S^n \rangle$ where $\sigma_a(S) = 0$ and S is cyclically reduced. If R is not in the derived subgroup then clearly such an algorithm will keep R out of the derived subgroup (the abelianisation contains torsion). Further, such an algorithm determines whether or not R is primitive, as if S is primitive and cyclically reduced with $\sigma_a(S) = 0$ and $\sigma_b(S) \neq 0$ then by Proposition 1.3 we must have that $S = b^\epsilon$. On the other hand, if after reduction we have that $S = b^\epsilon$ then R must have been primitive by the solution to the isomorphism problem for two-generator one-relator groups with torsion, as given in [13]. Note that if $S = b^\epsilon$ we can re-write it as $S = b$. Thus,

Lemma 3.1. *Let $G = \langle a, b; R^n \rangle$ with $n > 0$ such that the word R is not a proper power and not contained in $F(a, b)'$. Then there exists an algorithm to re-write R such that $G \cong \langle a, b; R^n \rangle$ with $\sigma_a(R) = 0$ and $\sigma_b(R) \neq 0$. If R is primitive then, after re-writing, $R = b$.*

Throughout this section, we assume $R \notin F(a, b)'$. Therefore, Lemma 3.1 provides an algorithm for taking $G = \langle a, b; R^n \rangle$ and yielding a presentation $G_0 = \langle a, b; S^n \rangle$, S cyclically reduced, such that $G \cong G_0$, $\sigma_a(S) = 0$ and $\sigma_b(S) \neq 0$. In Section 3.1, we prove that if R is non-primitive then $\text{Out}(G_0) \cong \text{Out}(G)$ is infinite if and only if S is contained in one of two specific subgroups of $F(a, b)$ if and only if S is fixed by one of two specific automorphisms of $F(a, b)$. This yields an algorithm for determining if $\text{Out}(G_0)$ is infinite or not. In Section 3.2 we prove that if R is non primitive then then the isomorphism class of $\text{Out}(G_0)$ can be discovered by checking which of only finitely many specific Nielsen transformations define automorphisms of G . Putting this all together yields an algorithm to determine $\text{Out}(G)$ when R is non-primitive and $R \notin F(a, b)'$.

3.1. The Infiniteness of $\text{Out}(G)$. In this subsection, we provide an algorithm to discover whether $\text{Out}(G)$ is infinite or not for a given $R \notin F(a, b)'$, and R not primitive. We also give an algorithm to determine $\text{Out}(G)$ if it is finite. We establish many of the ideas and results we use later on.

Letting L denote the group of tame automorphisms of some arbitrary two-generated group $G = \langle a, b; \mathbf{r} \rangle$,

Lemma 3.2. *Let $G = \langle a, b; \mathbf{r} \rangle$ such that $\sigma_a(R) = 0$ for all $R \in \mathbf{r}$ and $\mathbf{r} \not\subset F(a, b)'$. Then, if $\psi \in L$ there exists $\phi \in L$ such that*

$$\begin{aligned}\phi : a &\mapsto a^{\epsilon_0} b^k \\ b &\mapsto b^{\epsilon_1}\end{aligned}$$

and $\phi \equiv \psi \text{ mod } \text{Inn}(G)$.

Proof. Note that if

$$\begin{aligned}\phi : a &\mapsto a^{\epsilon_0} b^k \\ b &\mapsto b^{\epsilon_1}\end{aligned}$$

is a homomorphism then it is also an automorphism, as it is clearly surjective and right-invertible, so it is a bijection.

As $\psi \in L$ we can assume ψ is a Nielsen transformation of (a, b) . Write $a\psi := A$ and $b\psi := B$, and so (A, B) is a primitive pair of $F(a, b)$.

Let $\pi : G \rightarrow G^{ab}$ be the abelianisation map. The abelianisation has presentation $G^{ab} = \langle x, y; y^m, [x, y] \rangle$ because $\sigma_a(R) = 0$ for all $R \in \mathbf{r}$ but $\mathbf{r} \not\subset F(a, b)'$ ($x := a\pi$ and $y := b\pi$ while $m := \gcd(\sigma_b(R) : R \in \mathbf{r})$ is some integer greater than zero). Let $x^i y^\alpha := A\pi$ and let $x^j y^\beta := B\pi$. Then as G' is characteristic in G , automorphisms of G define automorphisms of $G^{ab} = G/G'$, so $B\pi$ has order $m \neq 0$. Thus,

$$(x^j y^\beta)^m = 1 \Rightarrow x^{mj} y^{m\beta} = 1 \Rightarrow x^{mj} = 1$$

so $mj = 0$ as x has infinite order in G^{ab} . Thus, $j = 0$ and so $B\pi = b^\beta$.

Therefore, $\sigma_a(B) = 0$.

By Proposition 1.3, the Nielsen transformation ψ corresponds to the matrix

$$\begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ 0 & \sigma_b(B) \end{pmatrix}$$

of $GL(2, \mathbb{Z})$ and so $|\sigma_a(A)| = 1 = |\sigma_b(B)|$. Taking $k := \sigma_b(A)$, $\epsilon_0 := \sigma_a(A)$ and $\epsilon_1 := \sigma_b(B)$, the Nielsen transformation,

$$\begin{aligned} \phi : a &\mapsto a^{\epsilon_0} b^k \\ b &\mapsto b^{\epsilon_1} \end{aligned}$$

also corresponds to this matrix. Now, if two Nielsen transformations are equal mod $\text{Inn}(F(a, b))$ they must also be equal mod $\text{Inn}(G)$, and so we are done. \square

Writing $K_0 := \{\phi : a\phi = a^{\epsilon_0} b^k, b\phi = b^{\epsilon_1}\} \subseteq \text{Aut}(F(a, b))$ and

$$K = \langle K_0, \text{Inn}(F(a, b)) \rangle \leq \text{Aut}(G)$$

we have

$$\frac{K}{\text{Inn}(F(a, b))} = \{\phi \text{Inn}(F(a, b)) : a\phi = a^{\epsilon_0} b^k, b\phi = b^{\epsilon_1}\}.$$

Lemma 3.2 means we have Figure 4, where H is as in Figure 1,

$$\begin{array}{ccccc} & & \text{Out}(F(a, b)) & & \\ & \nearrow & \uparrow & & \\ \frac{K}{\text{Inn}(F(a, b))} & \longleftarrow & \frac{H}{\text{Inn}(F(a, b))} & \xrightarrow{\theta} & \frac{L}{\text{Inn}(G)} \end{array}$$

FIGURE 4

Note, however, that not every Nielsen transformations of the above form need occur as automorphisms of G (that is, H can be a proper

subgroup of K). Indeed, we shall prove later that there are some two-generator one-relator groups where $H/\text{Inn}(F(a, b))$ is finite, and some where $H/\text{Inn}(F(a, b))$ (and so $\text{Out}(G)$ is trivial if the relator is a proper power).

We shall later prove that $K/\text{Inn}(F(a, b))$ is virtually cyclic, and so $L/\text{Inn}(F(a, b))$ is virtually cyclic, and we shall also later prove that every two-generated group with $G^{ab} \cong \mathbb{Z} \times C_m$ has some presentation which satisfies the conditions of Lemma 3.2.

Let us switch back to the one-relator case. So, take $\mathbf{r} = \{R^n\}$ in Lemma 3.2 where $n > 1$ and $R \notin F(a, b)'$. That is, $G = \langle a, b; R^n \rangle$. If R is non-primitive then by Proposition 1.1 we have that $L/\text{Inn}(G) = \text{Out}(G)$. In this case, we prove that θ has trivial kernel. On the other hand, if R is primitive then the map

$$\begin{aligned} \phi : a &\mapsto ab^n \\ b &\mapsto b \end{aligned}$$

is in the kernel of θ , so clearly $\text{Ker}(\theta)$ cannot be trivial.

To overcome this problem when R is primitive we notice that if two elements of K_0 define different automorphisms of G then they lie in different cosets of $\text{Aut}(G)/\text{Inn}(G)$. This is also what we prove when R is not primitive, as the pairs $(a^{\epsilon_0}b^i, b^{\epsilon_1})$ and $(a^{\epsilon'_0}b^j, b^{\epsilon'_1})$ are equal in the automorphism group only if $G = \langle a, b; b^n \rangle$ and $i = j \pmod n$. This is because if $a^{\epsilon_0}b^i = a^{\epsilon'_0}b^j$ then we have either $b^{i-j} = 1$, and so we can apply the Freiheitssatz, or $a^{-2}b^{i-j} =_G 1$, which can never happen as a has infinite order in the abelianisation whilst b has finite order.

Once we have proven that if two elements of K_0 define different automorphisms of G then they lie in different cosets of $\text{Aut}(G)/\text{Inn}(G)$, we can combine it with the fact that some subset of K_0 forms a transversal for $\text{Aut}(G)/\text{Inn}(G)$. Our algorithm to determine $\text{Out}(G)$ consists of finding this transversal, and the isomorphism class of $\text{Out}(G)$ is got relatively easily because if two elements of K_0 define different automorphisms of G then they lie in different cosets of $\text{Aut}(G)/\text{Inn}(G)$, and so we multiply the elements of K_0 which define automorphisms of G together in $\text{Out}(F(a, b))$. In the case of R not being primitive we need add no more relations, while in the case of R being primitive we have to work slightly harder.

The following Lemma is used to prove that no two elements of S are equal mod $\text{Inn}(G)$ unless they define the same automorphism of G , and is presumably known. We provide a proof, for completeness. First, however, note that a word of the form $a^{-i}b^ja^ib^k$ can never be a proper power for $i, j, k \neq 0$ in $F(a, b)$. Suppose otherwise, then $a^{-i}b^ja^ib^k \equiv S^n$,

$n > 1$, and S must begin with an a^ϵ and end in a $b^{\epsilon'}$. This means that no free cancellation happens when forming the word S^n , and so there exist two a^ϵ -terms in S^n which contain a $b^{\epsilon'}$ -term between them. However, as $S^n = a^{-i}b^ja^ib^k$, this is clearly a contradiction. We will use this observation a number of times in the proofs of the following two lemmata.

Lemma 3.3. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and $\sigma_a(R) = 0$. Then the centraliser of b in G consists of precisely the powers of b ,*

$$C_G(b) = \langle b \rangle.$$

Proof. We can, without loss of generality, assume that R is cyclically reduced.

By Proposition II.5.30 (p110) of [11] we have that $C_G(b)$ is cyclic, $C_G(b) = \langle U \rangle$ for some $U \in F(a, b)$. We wish to prove that $U = b^\epsilon$.

As $b \in C_G(b)$, $b = U^i$ for some $i \in \mathbb{C} \setminus \{0\}$ and so we can take U such that (a, U) is a primitive pair of $F(a, b)$. By Proposition 1.3 we have that there exists some $k \in \mathbb{Z}$ and $w \in F(a, b)$ with $w^{-1}aw \equiv a$ and $w^{-1}Uw \equiv b^\epsilon a^k$. As we have equality as words, $w \in \langle a \rangle$. Thus, $U \equiv a^{j_1}b^\epsilon a^{j_2}$, so $b =_G (a^{j_1}b^\epsilon a^{j_2})^i$.

Clearly it must hold that the images of b and $(a^{j_1}b^\epsilon a^{j_2})^i$ must be equal in the abelianisation of G . The abelianisation of G has presentation $\langle x, y; y^m, [x, y] \rangle$ with $m = n\sigma_b(R)$, and with $a \mapsto x$ and $b \mapsto y$. Thus, $y = x^{(j_1+j_2)i}y^{\epsilon i}$, and so $j_1 = -j_2$.

Therefore, we have that if $C_G(b) = \langle U \rangle$ then $U = a^{-j}b^\epsilon a^j$. As $U^i = b$ we have that $a^{-j}b^{\epsilon i}a^jb^{-1} = 1$, and applying the Newman-Gurevich Spelling Theorem (Proposition 1.2) we have either that $a^{-j}b^{\epsilon i}a^jb^{-1}$ must have two disjoint subwords, V and W , which are cyclic shifts of R or R^{-1} or that $a^{-j}b^{\epsilon i}a^jb^{-1} \equiv S^n$ where S is some cyclic shift of R or R^{-1} . The former case cannot happen, because $\sigma_a(V) = 0 = \sigma_a(W)$, so as R is cyclically reduced we have that $V^{n-1}V_0$ is a subword of $b^{\epsilon i}$ and $W^{n-1}W_0$ is a subword of b^{-1} . This is clearly a contradiction. Therefore, the latter case must happen: $a^{-j}b^{\epsilon i}a^jb^{-1} \equiv S^n$ where S is some cyclic shift of R or R^{-1} . However, $a^{-j}b^{\epsilon i}a^jb^{-1}$ is not a proper power of any word in $F(a, b)$ for $i, j \neq 0$. As $i \neq 0$, $j = 0$ and so $U = b^\epsilon$, as required. \square

We now prove that no two elements of S are equal mod $\text{Inn}(G)$ unless they define the same automorphism of G . Therefore, if $G = \langle a, b; R^n \rangle$, R is not primitive and $n > 1$, then θ is trivial and so $\text{Out}(G)$ embeds into $\text{Out}(F(a, b))$ in a canonical way.

Lemma 3.4. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and $\sigma_a(R) = 0$. Then if*

$$\begin{aligned}\phi_1 : a &\mapsto a^{\epsilon_0} b^i \\ b &\mapsto b^{\epsilon_1}\end{aligned}$$

and

$$\begin{aligned}\phi_2 : a &\mapsto a^{\epsilon'_0} b^j \\ b &\mapsto b^{\epsilon'_1}\end{aligned}$$

are two automorphisms of G ($\epsilon_0 \neq \epsilon'_0$, or $\epsilon_1 \neq \epsilon'_1$, or $i \neq j$) which are non-equal in $\text{Aut}(G)$ then they lie in different cosets of $\text{Aut}(G)/\text{Inn}(G)$.

Proof. Note that it is sufficient to prove that if $\phi \in \text{Inn}(G)$ with $\phi : a \mapsto a^{\epsilon_0} b^k, b \mapsto b^{\epsilon_1}$ then $a =_G a^{\epsilon_0} b^k$ and $b =_G b^{\epsilon_1}$. That is, it is sufficient to prove that the epimorphism θ from Figure 2 has trivial kernel.

So, let $\phi : a \mapsto a^{\epsilon_0} b^k, b \mapsto b^{\epsilon_1}$ with either $\epsilon_0 \neq 1$, or $b^{\epsilon_1} \neq_G b$ or $b^k \neq_G 1$, and assume $\phi \in \text{Inn}(G)$. Therefore, there exists $W \in G \setminus \{1\}$ such that $a^W = a^{\epsilon_0} b^k$ and $b^W = b^{\epsilon_1}$ but W contains no more than half of R^n (recall that if $W =_G 1$, so $a =_G a^{\epsilon_0} b^k$ and $b =_G b^{\epsilon_1}$, then we necessarily have that $\epsilon_0 = 1, b^{\epsilon_1} =_G b$ and $b^k =_G 1$, so we can assume $W \neq 1$).

If $a^{\pm 1} \not\leq R$ then G is the free product of $\langle a \rangle \cong \mathbb{Z}$ and $\langle b \rangle \cong C_n$, the cyclic group of order n . However, there is then no such $W \in G \setminus \{1\}$ such that $a^W = a^{\epsilon_0} b^k$ and $b^W = b^{\epsilon_1}$, as required.

Therefore, we have two cases, $\epsilon_1 = 1$ and $\epsilon_1 = -1$, and we can assume $a^{\pm 1} \leq R$.

We begin by looking at the case of $\epsilon_1 = -1$: If this holds, then $W^{-1}bW = b^{-1}$, so $W^2 \in C_G(b)$. Thus, $W^2 = b^i$ for some $i \in \mathbb{Z}$ by Lemma 3.3. Now, as $W^{-1}bW = b^{-1}$ we have that $bWb = W$, so $b^i W b^i = W$, and so $W^4 = 1$ (substituting in $b^i = W^2$). Thus, $b^{2i} = 1$ and so either $R = b$ or $R = b^{-1}$ (by applying the Freiheitssatz to the fact that b has finite order, and because we are assuming R is cyclically reduced). However, this is a contradiction as we are assuming $a^{\pm 1} \leq R$. Thus, the case of $\epsilon_1 = -1$ cannot happen.

We now turn to the case of $\epsilon_1 = 1$, and we shall use the fact that $W = b^i$ for some $i \in \mathbb{Z} \setminus \{0\}$, by Lemma 3.3, as $b^W = b$.

Looking at the abelianisation of G , $G^{ab} = \langle x, y; y^m, [x, y] \rangle$ where $a \mapsto x$ and $b \mapsto y$, we see that if some word $U =_G 1$ then the image of U under the abelianisation map must contain no x -terms, because x has infinite order in G^{ab} . Thus, if $U =_G 1$ then $\sigma_a(U) = 0$.

Recall that $a^{\pm 1} \leq R$. Now, $b^{-i} a b^i = a^{\epsilon_0} b^k$, so $a^{\epsilon_0} b^{k-i} a^{-1} b^i = 1$. Thus, $\sigma_a(a^{\epsilon_0} b^{k-i} a^{-1} b^i) = 0$, so $\epsilon_0 = 1$. If $i = k$ then we have that $b^i = 1$, and so either $R = b$ or $R = b^{-1}$ (by applying the Freiheitssatz to the fact

that b has finite order, and because we are assuming R is cyclically reduced). This contradicts the fact that $a^{\pm 1} \leq R$. Thus, $i \neq k$.

As $a^{\pm 1} \leq R$, we can apply the Newman-Gurevich spelling theorem to get that either $ab^{k-i}a^{-1}b^i$ is a cyclic shift of R^n or R^{-n} , or there exists two disjoint subwords $S^{n-1}S_0$ and $T^{n-1}T_0$, where $a^{\pm 1} \leq S, S_0, T, T_0$. However, neither case can happen: The four words S, S_0, T , and T_0 are disjoint and each contain an $a^{\pm 1}$ -term, but there are only 2 occurrences of $a^{\pm 1}$ in $a^{\epsilon_0}b^{k-i}a^{-1}b^i$ so we have a contradiction. On the other hand, this word is not a proper power.

Thus, the case of $\epsilon = 1$ cannot happen, and this completes the result. \square

By Lemma 3.2, every automorphism of G is conjugate to an automorphism of the form

$$\begin{aligned}\phi : a &\mapsto a^{\epsilon_0}b^i \\ b &\mapsto b^{\epsilon_1}\end{aligned}$$

and by Lemma 3.4 these automorphisms can be taken as a transversal for $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$. The reader should note that this means the coset representatives can take four forms, depending on the choices for ϵ_0 and ϵ_1 . That is, every automorphism of G is of one of the following forms,

$$\begin{array}{ll}\alpha_i : a \mapsto a^{-1}b^i & \beta_i : a \mapsto ab^i \\ b \mapsto b & b \mapsto b^{-1} \\ \zeta_i : a \mapsto a^{-1}b^i & \delta_i : a \mapsto ab^i \\ b \mapsto b^{-1} & b \mapsto b\end{array}$$

and we shall use the labels $\alpha_i, \beta_i, \zeta_i$, and δ_i in the rest of the section to refer to these forms.

It will be useful to know how these functions compose modulo the inner automorphisms. We give some of the compositions below,

$$\begin{aligned}\beta_i\beta_j &= \delta_{j-i} \text{ mod Inn}(G) \\ \alpha_i\alpha_j &= \delta_{i-j} \text{ mod Inn}(G) \\ \beta_i\alpha_j &= \zeta_{i+j} \text{ mod Inn}(G) \\ \alpha_j\beta_i &= \zeta_{-i-j} \text{ mod Inn}(G) \\ \zeta_i^2 &= \delta_{-2i} \text{ mod Inn}(G)\end{aligned}$$

To see these,

- (1) For the first equality, $a\beta_i\beta_j = (ab^i)\beta_j = ab^{j-i}$ and $b\beta_i\beta_j = b^{-1}\beta_j = b$ while $a\delta_{j-i} = ab^{j-i}$ and $b\delta_{j-i} = b$, as required.
- (2) For the second equality, $a\alpha_i\alpha_j = (a^{-1}b^i)\alpha_j = b^{-j}ab^i$ and $b\alpha_i\alpha_j = b\alpha_j = b$ while $a\delta_{i-j} = ab^{i-j}$ and $b\delta_{i-j} = b$. As $b^j(b^{-j}ab^i)b^{-j} = ab^{i-j}$ and $b^jbb^{-j} = b$ we have that $\alpha_i\alpha_j = \delta_{i-j} \bmod \text{Inn}(G)$ and so we get the required result.
- (3) For the third equality, $a\beta_i\alpha_j = (ab^i)\alpha_j = a^{-1}b^{i+j}$ and $b\beta_i\alpha_j = b^{-1}\alpha_j = b^{-1}$ while $a\zeta_{i+j} = a^{-1}b^{i+j}$ and $b\zeta_{i+j} = b^{-1}$, as required.
- (4) For the fourth equality, $a\alpha_j\beta_i = (a^{-1}b^j)\beta_i = b^{-i}a^{-1}b^{-j}$ and $b\alpha_j\beta_i = b\beta_i = b^{-1}$ while $a\zeta_{-i-j} = a^{-1}b^{-i-j}$ and $b\zeta_{-i-j} = b^{-1}$. As in the second equality, we can apply the automorphism corresponding to conjugation by b^i to get that $\alpha_j\beta_i = \zeta_{-i-j} \bmod \text{Inn}(G)$, as required.
- (5) For the fifth equality, $a\zeta_i^2 = (a^{-1}b^i)\zeta_i = b^{-i}ab^{-i}$ and $b\zeta_i^2 = b^{-1}\zeta_i = b$ while $a\delta_{-2i} = ab^{-2i}$ and $b\delta_{-2i} = b$. As in the second and fourth equalities we can apply the automorphism corresponding to conjugation by b^{-i} to get that $\zeta_i^2 = \delta_{-2i} \bmod \text{Inn}(G)$, as required.

Theorem 3.5. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$, R not primitive, $\sigma_a(R) = 0$, and $\sigma_b(R) \neq 0$. Then $\text{Out}(G)$ is either infinite, $C_2 \times C_2$, C_2 or trivial. If $\text{Out}(G)$ is infinite then there exists $k \in \mathbb{Z}$ such that $\delta_k \in \text{Aut}(G)$.*

Proof. Along with the above equalities, we will use the fact that the coset of δ_i has infinite order in $\text{Out}(G)$ for all $i \in \mathbb{Z} \setminus \{0\}$. This holds, as if δ_i has finite order $\bmod \text{Inn}(G)$ there exists $k \in \mathbb{Z} \setminus \{1\}$ such that $\delta_i = \delta_i^k \bmod \text{Inn}(G)$; as $\delta_j = \delta_1^j$ we have that $\delta_i = \delta_i^k = \delta_{ik}$. This contradicts the fact that all the automorphisms of the form $a \mapsto a^{\epsilon_0}b^i$, $b \mapsto b^{\epsilon_1}$ lie in different cosets of $\text{Out}(G)$ (Lemma 3.4).

Now, noting that $\alpha_i\beta_{-i} = \zeta_0 = \beta_{-i}\alpha_i$, if α_i and β_{-i} are automorphisms of G then they, $\bmod \text{Inn}(G)$, generate a subgroup of $\text{Out}(G)$ isomorphic to $C_2 \times C_2$; they commute and each has order 2. If α_i , β_{-i} , and ζ_0 are the only non-trivial automorphisms up to conjugacy then $\text{Out}(G) \cong C_2 \times C_2$. Similarly, if one of α_i , β_j or ζ_0 defines an automorphism of G and there are no other non-trivial automorphisms modulo the inner automorphisms then $\text{Out}(G) \cong C_2$.

If $\zeta_i \in \text{Aut}(G)$ with $i \neq 0$ then $\delta_{-2i} \in \text{Aut}(G)$ and so $\text{Out}(G)$ is infinite.

If α_i and $\alpha_j \in \text{Aut}(G)$ with $i \neq j$ then $\delta_{i-j} \in \text{Aut}(G)$ and so $\text{Out}(G)$ is infinite.

If β_i and $\beta_j \in \text{Aut}(G)$ with $i \neq j$ then $\delta_{j-i} \in \text{Aut}(G)$ and so $\text{Out}(G)$ is infinite.

If α_i and $\beta_j \in \text{Aut}(G)$ with $i + j \neq 0$ then $\zeta_{i+j} \in \text{Aut}(G)$, so $\delta_{-2(i+j)} \in \text{Aut}(G)$ and so $\text{Out}(G)$ is infinite.

Thus, if $\text{Out}(G)$ is finite,

$$\begin{aligned} \text{Out}(G) &= \langle 1 \rangle \\ \text{or } \text{Out}(G) &= \langle \alpha_i \text{Inn}(G) \rangle \cong C_2 \\ \text{or } \text{Out}(G) &= \langle \beta_i \text{Inn}(G) \rangle \cong C_2 \\ \text{or } \text{Out}(G) &= \langle \zeta_0 \text{Inn}(G) \rangle \cong C_2 \\ \text{or } \text{Out}(G) &= \langle \alpha_i \text{Inn}(G), \beta_{-i} \text{Inn}(G) \rangle \cong C_2 \times C_2 \end{aligned}$$

Otherwise, $\delta_k \in \text{Aut}(G)$ for some $k \in \mathbb{Z}$, as required. \square

Note that the above Theorem says that $\text{Out}(G)$ is infinite if and only if there exists some $k \in \mathbb{Z}$ such that $\delta_k \in \text{Aut}(G)$.

We now wish to find out when the Nielsen transformations α_k , β_k and δ_k are in $\text{Aut}(G)$ for a given $k \in \mathbb{Z}$. We prove that $\delta_1 \in \text{Aut}(G)$ if and only if $\text{Out}(G)$ is infinite, and, further, that δ_1 fixes R or a cyclic shift of R if and only if $\text{Out}(G)$ is infinite, where $\sigma_a(R) = 0$ and $\sigma_b(R) \neq 0$. We also prove that if $\text{Out}(G)$ is finite then one can find a finite set of integers A (resp. B) such that α_k (resp. β_k) can be in $\text{Aut}(G)$ only if $k \in A$ (resp. $k \in B$). To prove these, we need the following lemma,

Lemma 3.6. *Let ϕ_k be the Nielsen transformation,*

$$\begin{aligned} \phi_k : a &\mapsto a^{\epsilon_0} b^k \\ &b \mapsto b^{\epsilon_1} \end{aligned}$$

and let W be an arbitrary, freely reduced word in $F(a, b)$. Then,

- (1) *If W begins in a , $W\phi_k$ begins in a^{ϵ_0} ,*
- (2) *If W begins in a^{-1} , $W\phi_k$ begins in $b^{-k}a^{-\epsilon_0}$,*
- (3) *If W ends in a^{-1} , $W\delta_k$ ends in $a^{-\epsilon_0}$,*
- (4) *If W ends in a , $W\delta_k$ ends in $a^{\epsilon_0}b^k$.*

Proof. Note that once we have proven (1) and (2) then (3) and (4) follow immediately, by looking at W^{-1} . To prove (1) and (2) we assume that $W = a^\epsilon \overline{W}$ is a word starting with an a -term and induct on the number of a -terms in the word W .

If W contains one a -term then $W = a^\epsilon b^i$ so $W\phi_k = a^{\epsilon_0} b^{\epsilon_1 i + k}$ if $\epsilon = 1$ while $W\phi_k = b^{-k} a^{-\epsilon_0} b^{\epsilon_1 i}$ if $\epsilon = -1$, as required.

Assume the result holds for all words beginning with an a -term and containing n a -terms, and let W be a word containing $n+1$ a -terms and beginning with an a -term. Then $W = a^\epsilon b^i a^{\epsilon'} \overline{W}$ where $a^{\epsilon'} \overline{W}$ satisfies the induction hypothesis and $i \neq 0$ if $\epsilon + \epsilon' = 0$. We thus have four cases to consider,

- $\epsilon = 1, \epsilon' = 1$: $W\phi_k = (ab^i)\phi_k(a\overline{W})\phi_k = a^{\epsilon_0}(b^{\epsilon_1 i+k}a^{\epsilon_0}\widehat{W})$,
- $\epsilon = 1, \epsilon' = -1$: $W\phi_k = (ab^i)\phi_k(a^{-1}\overline{W})\phi_k = a^{\epsilon_0}(b^{\epsilon_1 i}a^{-\epsilon_0}\widehat{W})$,
- $\epsilon = -1, \epsilon' = 1$: $W\phi_k = (a^{-1}b^i)\phi_k(a\overline{W})\phi_k = b^{-k}a^{-\epsilon_0}(b^{\epsilon_1 i}a^{\epsilon_0}\widehat{W})$,
- $\epsilon = -1, \epsilon' = -1$: $W\phi_k = (a^{-1}b^i)\phi_k(a^{-1}\overline{W})\phi_k = b^{-k}a^{-\epsilon_0}(b^{\epsilon_1 i-k}a^{-\epsilon_0}\widehat{W})$,

as required. \square

Define \min_+ to be the least integer such that $ab^i a$ is a subword of some cyclic shift of R^n and define \max_+ to be the greatest such integer. Further, define \min_- to be the least integer such that $a^{-1}b^i a^{-1}$ is a subword of some cyclic shift of R^n and define \max_- to be the greatest such integer. We then have the following important lemma, which gives us an algorithm to calculate $\text{Out}(G)$ if $\text{Out}(G)$ is finite, and classifies the words R (with $\sigma_a(R) = 0$, $\sigma_b(R) \neq 0$ and R not primitive) such that $\text{Out}(G)$ is infinite where $G = \langle a, b; R^n \rangle$, $n > 1$.

Lemma 3.7. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$, R not primitive, and $\sigma_a(R) = 0$ but $\sigma_b(R) \neq 0$. If $a^{\epsilon'} b^i a^{\epsilon'}$ is a subword of some cyclic shift of R^n for $i \in \mathbb{Z}$ then,*

- $\alpha_k \in \text{Aut}(G)$ only if $k = -(\max_+ + \min_+)$, or
 $\min_- - \min_+ \leq k \leq \max_- - \max_+$,
- $\beta_k \in \text{Aut}(G)$ only if $k = \min_+ + \max_+$, or
 $\max_+ - \max_- \leq k \leq \min_+ - \min_-$,
- $\delta_k \notin \text{Aut}(G)$ for $k \neq 0$.

Proof. Note that if $a^{\epsilon'} b^i a^{\epsilon'}$ is a subword of some cyclic shift of R^n then there exists some $j \in \mathbb{Z}$ such that $a^{-\epsilon'} b^j a^{-\epsilon'}$ is a subword of some cyclic shift of R^n . This is because $\sigma_a(R^n) = 0$. Thus, we can assume $\epsilon' = 1$.

Our observations about α_k and β_k are similar. What we prove about δ_k is more involved.

To prove the results, we start by using the Newman-Gurevich Spelling Theorem. If $ab^i a \leq R^n$ then for all $\phi \in \text{Aut}(G)$ there exists $\epsilon \in \{1, -1\}$ such that $a^\epsilon b^i a^\epsilon \leq R^n \phi$ for all $\phi \in \text{Aut}(G)$. Note that we have that $a^\epsilon b^i a^\epsilon$ is a subword of $R^n \phi$, not just a subword of a cyclic shift. This is because we have that $S^{n-1} S_0 \leq R^n \phi$, S a cyclic shift of R or R^{-1} , and either $a^\epsilon b^i a^\epsilon \leq S$ or $S \equiv b^{\epsilon j} a^\epsilon \bar{S} a^\epsilon b^{\epsilon k}$, $i = j + k$, and in each case $a^\epsilon b^i a^\epsilon \leq S S_0$, as required.

We prove the following:

- If $\alpha_k \in \text{Aut}(G)$ and $ab^i a$ is a subword of some cyclic shift of R^n then $a^{-\epsilon} b^{(i+k)\epsilon} a^{-\epsilon}$ is a subword of some cyclic shift of R^n .
- If $\beta_k \in \text{Aut}(G)$ and $ab^i a$ is a subword of some cyclic shift of R^n then $a^\epsilon b^{(k-i)\epsilon} a^\epsilon$ is a subword of some cyclic shift of R^n .

- If $\delta_k \in \text{Aut}(G)$ and $ab^i a$ is a subword of some cyclic shift of R^n then either $ab^{i-k} a$ or $ab^{i-2k} a$ is a subword of some cyclic shift of R^n .

These results prove the lemma. To see that they prove the lemma, note that i can take any value between \min_+ and \max_+ , $\min_+ \leq i \leq \max_+$, so:

For the α_k case, either $\epsilon = -1$ and $\min_+ \leq -i - k \leq \max_+$, and so substituting in $i = \min_+$ and $i = \max_+$ we get that $-(\min_+ + \max_+) \leq k \leq -(\min_+ + \max_+)$ as required, or $\epsilon = -1$ and we have $\min_- \leq i + k \leq \max_-$, and so substituting in $i = \min_+$ and $i = \max_+$ we get two inequalities,

$$\min_- - \max_+ \leq k \leq \max_- - \max_+$$

$$\min_- - \min_+ \leq k \leq \max_- - \min_+,$$

and combining these we see that $\min_- - \min_+ \leq k \leq \max_- - \max_+$ as required.

For the β_k case, either $\epsilon = 1$ and $\min_+ \leq k - i \leq \max_+$, and so substituting in $i = \min_+$ and $i = \max_+$ we get that $\min_+ + \max_+ \leq k \leq \min_+ + \max_+$ as required, or $\epsilon = -1$ and we have $\min_- \leq i - k \leq \max_-$, and so substituting in $i = \min_+$ and $i = \max_+$ we get two inequalities,

$$\min_+ - \min_- \geq k \geq \min_+ - \max_-$$

$$\max_+ - \min_- \geq k \geq \max_+ - \max_-,$$

and combining these we see that $\max_+ - \max_- \leq k \leq \min_+ - \min_-$ as required.

For the δ_k case, $\delta_k^{-1} = \delta_{-k}$ so we can assume $k > 0$, thus taking i to be the least integer such that $ab^i a$ is a subword of some cyclic shift of R^n (so, $i = \min_+$) we have a contradiction.

We now prove the three statements.

Recall that if $ab^i a \leq R^n$ then for all $\phi \in \text{Aut}(G)$ there exists $\epsilon \in \{1, -1\}$ such that $a^\epsilon b^{\epsilon i} a^\epsilon \leq R^n \phi$ for all $\phi \in \text{Aut}(G)$.

Writing γ_g for the automorphism inducing conjugation by g , so $a\gamma_g = g^{-1}ag$ and $b\gamma_g = g^{-1}bg$, we investigate the three cases:

- $\alpha_k \in \text{Aut}(G)$: Assume $a^\epsilon b^{\epsilon i} a^\epsilon$ is a subword of $R^n \alpha_k$, so $R^n \alpha_k \equiv U a^\epsilon b^{\epsilon i} a^\epsilon V$. Then as $\alpha_k^{-1} = \alpha_k \gamma_{b^{-k}}$ we can apply Lemma 3.6 to this, and so

$$\begin{aligned} R^n &\equiv ((U a^\epsilon) \alpha_k (b^{\epsilon i} \alpha_k) (a^\epsilon V) \alpha_k) \gamma_{b^{-k}} \\ &\equiv (U' a^{-\epsilon} b^{(i+k)\epsilon} a^{-\epsilon} V') \gamma_{b^{-k}} \\ &\equiv b^k U' a^{-\epsilon} b^{(i+k)\epsilon} a^{-\epsilon} V' b^{-k} \end{aligned}$$

as required.

- $\beta_k \in \text{Aut}(G)$: Assume $a^\epsilon b^{\epsilon i} a^\epsilon$ is a subword of $R^n \beta_k$, so $R^n \beta_k \equiv U a^\epsilon b^{\epsilon i} a^\epsilon V$. Then as $\beta_k^{-1} = \beta_k$ we can apply Lemma 3.6 to this, and so

$$\begin{aligned} R^n &\equiv (U a^\epsilon) \beta_k (b^{\epsilon i} \beta_k) (a^\epsilon V) \beta_k \\ &\equiv U' a^\epsilon b^{(k-i)\epsilon} a^\epsilon V' \end{aligned}$$

as required.

- $\delta_k \in \text{Aut}(G)$: We have two cases, $\epsilon = 1$ and $\epsilon = -1$,
 - (1) Assume $\epsilon = 1$. That is, $ab^i a$ is a subword of $R^n \delta_k$, so $R^n \delta_k \equiv U a b^i a V$. Then as $\delta_k^{-1} = \delta_{-k}$ we can apply Lemma 3.6 to this, and so

$$\begin{aligned} R^n &\equiv (U a) \delta_{-k} (b^i \delta_{-k}) (a V) \delta_{-k} \\ &\equiv U' a b^{i-k} a V' \end{aligned}$$

as required.

- (2) Assume $\epsilon = -1$. That is, $a^{-1} b^{-i} a^{-1}$ is a subword of $R^n \delta_k$ but $ab^i a$ is not, so $R^n \delta_k \equiv U a^{-1} b^{-i} a^{-1} V$. Then as $\delta_k^{-1} = \delta_{-k}$ we can apply Lemma 3.6 to this, and so

$$\begin{aligned} R^n &\equiv (U a^{-1}) \delta_{-k} (b^{-i} \delta_{-k}) (a^{-1} V) \delta_{-k} \\ &\equiv U' a^{-1} b^{k-i} a^{-1} V'. \end{aligned}$$

Therefore, $a^{-1} b^{k-i} a^{-1}$ is a subword of a cyclic shift of R^n . Now, by the Newman-Gurevich Spelling Theorem there exists S a cyclic shift of R or R^{-1} such that $S^{n-1} S_0$ is a subword of $R^n \delta_k$. As $ab^i a$ is not a subword of $R^n \delta_k$ but is a subword of R^n we must have that S is a cyclic shift of R^{-1} , and so $ab^{i-k} a$ is a subword of $R^n \delta_k$. Therefore, $R^n \delta_k \equiv U_0 a b^{i-k} a V_0$. Then as $\delta_k^{-1} = \delta_{-k}$ we can apply Lemma 3.6 to this, and so

$$\begin{aligned} R^n &\equiv (U_0 a) \delta_{-k} (b^{i-k} \delta_{-k}) (a V_0) \delta_{-k} \\ &\equiv U'_0 a b^{i-2k} a V'_0 \end{aligned}$$

as required.

□

If we write $\delta := \delta_1$, we have the following theorem,

Theorem 3.8. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$, $R \notin F(a, b)$ and R not primitive. After re-writing R such that $\sigma_a(R) = 0$, the following are equivalent,*

- (1) $\text{Out}(G)$ is infinite,
- (2) $R \in \langle aba^{-1}, b \rangle \cup \langle a^{-1}ba, b \rangle$,
- (3) $\delta \in \text{Aut}(G)$.

Proof. (1 \Rightarrow 2) If $R \notin \langle aba^{-1}, b \rangle \cup \langle a^{-1}ba, b \rangle$ then $a^\epsilon b^i a^\epsilon$ is a subword of R^n for some $i \in \mathbb{Z}$, and so by Lemma 3.7 we have that $\text{Out}(G)$ is finite. Thus, if R is infinite then $R \in \langle aba^{-1}, b \rangle \cup \langle a^{-1}ba, b \rangle$, as required.

(2 \Rightarrow 3) Assume $R \in \langle aba^{-1}, b \rangle \cup \langle a^{-1}ba, b \rangle$. Without loss of generality we can assume $R \in \langle aba^{-1}, b \rangle$, as otherwise taking $S := aRa^{-1}$, S is in $\langle aba^{-1}, b \rangle$ and $\langle a, b; R^n \rangle \cong \langle a, b; S^n \rangle$. Now, R is mapped to itself freely under the map

$$\begin{aligned} \delta : a &\mapsto ab \\ b &\mapsto b \end{aligned}$$

so we have that δ is a homomorphism, and so an automorphism. Thus, $\delta \in \text{Aut}(G)$, as required.

(3 \Rightarrow 1) If $\delta \in \text{Aut}(G)$ then δ has infinite order mod $\text{Inn}(G)$ by Lemma 3.4, and so $\text{Out}(G)$ is infinite, as required. \square

Therefore, if R is not primitive there exists an algorithm for determining whether $\text{Out}(G)$ is infinite or not,

- As $R \notin F(a, b)'$, by Lemma 3.1 there exists an algorithm to find a word S in $F(a, b)$ such that $G \cong \langle a, b; S^n \rangle$ and $\sigma_a(S) = 0$ and $\sigma_b(S) \neq 0$.
- Calculate $S\delta$.
- If $S\delta \equiv S$ or $S\delta \equiv aSa^{-1}$ then $\text{Out}(G)$ is infinite.
- $\text{Out}(G)$ is finite otherwise.

On the other hand, if $\text{Out}(G)$ is finite there exists an algorithm for determining $\text{Out}(G)$,

- As $R \notin F(a, b)'$, by Lemma 3.1 there exists an algorithm to find a word S in $F(a, b)$ such that $G \cong \langle a, b; S^n \rangle$ and $\sigma_a(S) = 0$ and $\sigma_b(S) \neq 0$.
- Calculate $S\zeta_0$, then $\zeta_0 \in \text{Aut}(G)$ if and only if $S\zeta_0 =_G 1$.
- Calculate $S\alpha_k$ for all k such that $k = -(\max_+ + \min_+)$ or

$$\min_- - \min_+ \leq k \leq \max_- - \max_+$$

then $\alpha_k \in \text{Aut}(G)$ if and only if $S\alpha_k =_G 1$.

- Calculate $S\beta_k$ for all k such that $k = \max_+ + \min_+$ or

$$\max_+ - \max_- \leq k \leq \min_+ - \min_-$$

then $\beta_k \in \text{Aut}(G)$ if and only if $S\beta_k =_G 1$.

- If there exists some k such that α_k, β_{-k} and ζ_0 are all in $\text{Aut}(G)$ then

$$\text{Out}(G) \cong C_2 \times C_2$$

- Else if $\zeta_0 \in \text{Aut}(G)$ or if there exists some k in the above ranges such that α_k or $\beta_k \in \text{Aut}(G)$, then

$$\text{Out}(G) \cong C_2$$

- Else, $\text{Out}(G)$ is trivial.

Note that by Theorem 3.5 these are the only three possible isomorphism classes for $\text{Out}(G)$ if $\text{Out}(G)$ is finite, because if $\text{Out}(G)$ is finite one cannot have $\alpha_i, \alpha_j \in \text{Aut}(G)$ unless $i = j$, or $\beta_i, \beta_j \in \text{Aut}(G)$ unless $i = j$, or $\alpha_i, \beta_j \in \text{Aut}(G)$ unless $i = -j$, or $\zeta_i \in \text{Aut}(G)$ unless $i = 0$.

3.2. Investigating $\text{Out}(G)$ when $\text{Out}(G)$ is infinite. Assuming $R \notin F(a, b)'$, non-primitive, we have proven that if $\text{Out}(G)$ is finite then $\text{Out}(G)$ is either trivial, C_2 or $C_2 \times C_2$ (Theorem 3.5), and we have an algorithm to compute which one $\text{Out}(G)$ is. We also have an algorithm for deciding whether $\text{Out}(G)$ is infinite or not. It therefore remains only to show what $\text{Out}(G)$ looks like when it is infinite. This is not, however, too hard. Assume $\text{Out}(G)$ is infinite, so $\delta \in \text{Aut}(G)$, then one can view ζ_i as $\delta^{-i}\zeta$ where $\zeta := \zeta_0$, and so if $\zeta_i \in \text{Aut}(G)$ then so is ζ_j for all $j \in \mathbb{Z}$. Similarly, $\alpha_i = \delta^i\alpha$ and $\beta_i = \delta^{-i}\beta$, where $\alpha := \alpha_0$ and $\beta := \beta_0$, and so if $\alpha_i \in \text{Aut}(G)$ (resp. $\beta_i \in \text{Aut}(G)$) then so is α_j (resp. β_j) for all $j \in \mathbb{Z}$. Now, as $\alpha\beta = \zeta$ then if α and β are in $\text{Aut}(G)$ then so is ζ . Similarly, if ζ and α are then so is β and if ζ and β are then so is α . What this means is that we have five choices of generating set for $\text{Out}(G)$ if $\text{Out}(G)$ is infinite. We always have $\delta \in \text{Aut}(G)$, and,

- (1) $\alpha, \beta, \zeta \notin \text{Aut}(G)$,
- (2) $\alpha \in \text{Aut}(G)$ but $\beta, \zeta \notin \text{Aut}(G)$,
- (3) $\beta \in \text{Aut}(G)$ but $\alpha, \zeta \notin \text{Aut}(G)$,
- (4) $\zeta \in \text{Aut}(G)$ but $\alpha, \beta \notin \text{Aut}(G)$,
- (5) $\alpha, \beta, \zeta \in \text{Aut}(G)$.

Note that each of these possibilities occur;

- (1) If $R = aba^{-1}b^2ab^3a^{-1}b^4$ then $\alpha, \beta, \zeta \notin \text{Aut}(G)$,
- (2) If $R = aba^{-1}b^2ab^3a^{-1}bab^2a^{-1}b^3$ then $\alpha \in \text{Aut}(G)$ but $\beta, \zeta \notin \text{Aut}(G)$,
- (3) If $R = aba^{-1}b^2$ then $\beta \in \text{Aut}(G)$ but $\alpha, \zeta \notin \text{Aut}(G)$,

- (4) If $R = aba^{-1}b^2ab^2a^{-1}bab^3a^{-1}b^3$ then $\zeta \in \text{Aut}(G)$ but $\alpha, \beta \notin \text{Aut}(G)$,
- (5) If $R = aba^{-1}b$ then $\alpha, \beta, \zeta \in \text{Aut}(G)$.

Checking these results is an easy exercise, as none of α, β, ζ change the length of R , so for $\gamma \in \{\alpha, \beta, \zeta\}$, if $\gamma \in \text{Aut}(G)$ then $R\gamma$ is a cyclic shift of R or R^{-1} by the Newman-Gurevich Spelling Theorem.

3.2.1. *What does $\text{Out}(G)$ look like?* We wish to find what $\text{Out}(G)$ looks like in each of these five cases. If $\text{Out}(G) = \langle \delta \rangle$ then clearly $\text{Out}(G) \cong \mathbb{Z}$. Otherwise, the presentations are easily acquired as there is a normal form; every element is of the form $\delta^i \gamma$ with $\gamma \in \{\alpha, \beta, \zeta, e\}$. By Lemma 3.4, an element of this normal form is trivial modulo the inner automorphisms if and only if $i = 1$ and $\gamma = e$. This means that once we have added the relators to the group which get elements into this normal form (which we can work out as we have a representation for $\text{Out}(G)$ in terms of Nielsen transformations) we need add no more relators. The groups are,

- (1) $\alpha, \beta, \zeta \notin \text{Aut}(G)$, and so $\text{Out}(G) \cong \mathbb{Z}$,
- (2) $\alpha \in \text{Aut}(G)$ but $\beta, \zeta \notin \text{Aut}(G)$, and so

$$\text{Out}(G) \cong \langle \delta, \alpha; \alpha^2, \alpha\delta = \delta^{-1}\alpha \rangle \cong D_\infty$$

- (3) $\beta \in \text{Aut}(G)$ but $\alpha, \zeta \notin \text{Aut}(G)$, and so

$$\text{Out}(G) \cong \langle \delta, \beta; \beta^2, \beta\delta = \delta^{-1}\beta \rangle \cong D_\infty$$

- (4) $\zeta \in \text{Aut}(G)$ but $\alpha, \beta \notin \text{Aut}(G)$, and so

$$\text{Out}(G) \cong \langle \delta, \zeta; \zeta^2, [\delta, \zeta] \rangle \cong \mathbb{Z} \times C_2$$

- (5) $\alpha, \beta, \zeta \in \text{Aut}(G)$, and so we have the following relations

$$\alpha^2 = e, \beta^2 = e, \zeta^2 = e, \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \delta\zeta = \zeta\delta$$

$$\alpha\beta = \zeta, \alpha\zeta = \beta, \beta\alpha = \zeta, \beta\zeta = \alpha, \zeta\alpha = \beta, \zeta\beta = \alpha$$

and all of the relations in the second line follow from the relations

$$\alpha^2 = e, \beta^2 = e, \zeta^2 = e, \alpha\beta\zeta = e.$$

We therefore have that

$$\text{Out}(G) \cong \langle \alpha, \beta, \delta, \zeta; \alpha^2, \beta^2, \zeta^2, [\delta, \zeta], \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \alpha\beta\zeta \rangle.$$

Replacing β with $\alpha\zeta$, and following the Tietze transformations through we get

$$\begin{aligned} \text{Out}(G) &\cong \langle \alpha, \delta, \zeta; \alpha^2, \zeta^2, \alpha\delta\alpha = \delta^{-1}, [\alpha, \zeta], [\delta, \zeta] \rangle \\ &\cong D_\infty \times C_2 \end{aligned}$$

Note that in each case $\text{Out}(G)$ is a \mathbb{Z} - by - finite semidirect product. Further, note that this proves that, under the assumptions of this section, there exists an algorithm to find $\text{Out}(G)$ if $\text{Out}(G)$ is infinite,

- As $R \notin F(a, b)'$, by Lemma 3.1 there exists an algorithm to find a word S in $F(a, b)$ such that $G \cong \langle a, b; S^n \rangle$ and $\sigma_a(S) = 0$ and $\sigma_b(S) \neq 0$.
- Calculate $S\gamma$ for each $\gamma \in \{\alpha, \beta, \zeta\}$.
- $\gamma \in \text{Aut}(G)$ if and only if $S\gamma \equiv T$ or $S\gamma \equiv T^{-1}$, where T is a cyclic shift of S .
- The isomorphism class of $\text{Out}(G)$ is got by comparing which of these γ are in $\text{Aut}(G)$ with the above list.

4. $\text{Out}(G)$ FOR R PRIMITIVE

Let $G = \langle a, b; R^n \rangle$. If R is primitive then $G \cong \langle a, b; b^n \rangle$ by Lemma 3.1. In this section we prove that if R is primitive then

$$\text{Out}(G) \cong D_n \rtimes \text{Aut}(C_n),$$

a group of order $2n\varphi(n)$, where $\text{Aut}(C_n)$ commutes with the flip generator of D_n and acts on the rotations in the natural way as automorphisms of C_n (the rotations form a group isomorphic to C_n).

As R is primitive, G has $\frac{1}{2}\varphi(n)$ Nielsen equivalence classes, and by [13] the automorphisms $\psi_k : a \mapsto a, b \mapsto b^k$ sends (a, b) to a new Nielsen equivalence class if $k \not\equiv \pm 1 \pmod{n}$, where $0 < k < n$ and $\gcd(k, n) = 1$. Again by [13], $(a\psi_i, b\psi_i)$ is Nielsen equivalent to $(a\psi_j, b\psi_j)$ if and only if $i \equiv \pm j \pmod{n}$. Clearly the automorphisms ψ_k form the group $\text{Aut}(C_n)$.

On the other hand, the automorphisms which keep (a, b) in the same Nielsen equivalence class (the tame automorphisms) are the automorphisms of the form $a \mapsto a^{\epsilon_0} b^k, b \mapsto b^{\epsilon_1}$, $0 \leq k < n$, and these form the group $D_n \times C_2$, where the C_2 corresponds to the automorphism ψ_{n-1} .

We use Lemma 2.1 to pin these two groups together over the automorphism ψ_{n-1} , and this gives us the semidirect product in question.

Theorem 4.1. *If $G = \langle a, b; R^n \rangle$ R is a primitive element of $F(a, b)$ then*

$$\text{Out}(G) \cong D_n \rtimes \text{Aut}(C_n)$$

where $\text{Aut}(C_n)$ commutes with the flip generator of D_n and acts on the rotation generator in the natural way as automorphisms of C_n .

Proof. We can, by Lemma 3.1, assume that $G = \langle a, b; b^n \rangle$ with $n > 1$.

It is clear that every function of the form $\phi : a \mapsto a^{\epsilon_0} b^i, b \mapsto b^{\epsilon_1}$ is in $\text{Aut}(G)$, and Lemma 3.4 gives us that these are all non-equal mod $\text{Inn}(G)$ for $0 \leq i < n$, while Lemma 3.2 gives us that these are the only

automorphisms, modulo the inner automorphisms, which keep (a, b) in the same Nielsen equivalence class. Therefore, keeping the same notation as Section 3.2, α, β, δ and ζ are all in $\text{Aut}(G)$, are all non-equal mod $\text{Inn}(G)$, and every tame automorphism is of the form $\delta^i \gamma$ where $\gamma \in \{\alpha, \beta, \zeta, e\}$, modulo the inner automorphisms of G . Note that δ has order n in both $\text{Aut}(G)$ and $\text{Out}(G)$.

Thus, by Lemma 2.1, every automorphism is of the form $\delta^i \gamma \psi_c$ where $\gamma \in \{\alpha, \beta, \zeta, e\}$, modulo the inner automorphisms of G .

Note that there are only finitely many choices for ψ_c , by Proposition 1.1, and so we immediately have that $\text{Out}(G)$ is finite.

If $n = 2$ there is only one Nielsen equivalence class, so the ψ_c can be ignored and we get the required result of $D_2 \cong C_2 \times C_2$.

If $n > 2$ we now want to find out what the ψ_c are; what are the maps which take (a, b) to the other Nielsen equivalence classes. By [13], one can take these automorphisms to be the automorphisms $\psi_k : a \mapsto a, b \mapsto b^k$ where $\gcd(k, n) = 1$ and $0 < k < \frac{n}{2}$. Note that the generator (a, b^{n-k}) is in the same Nielsen equivalence class as (a, b^k) for all i , as (a, b^k) is mapped to (a, b^{n-k}) via the automorphism $\alpha : a \mapsto a, b \mapsto b^{-1}$, which is a Nielsen Transformation.

As inner automorphisms will keep the generating pair (a, b) in the same Nielsen equivalence class, none of the automorphisms ψ_k are equal modulo the inner automorphisms.

Now, note that if ϕ is some Nielsen transformation, $\phi : a \mapsto U_a(a, b), b \mapsto U_b(a, b)$ say, then $(a\phi\psi_k, b\phi\psi_k) = (a\psi_k, b\psi_k)\phi$. This holds because

$$\begin{aligned} (a\phi\psi_k, b\phi\psi_k) &= (U_a(a, b)\psi_k, U_b(a, b)\psi_k) \\ &= (U_a(a\psi_k, b\psi_k), U_b(a\psi_k, b\psi_k)) \\ &= (a\psi_k, b\psi_k)\phi \end{aligned}$$

as required.

So, we have that the maps $\phi : a \mapsto a^{\epsilon_0} b^i, b \mapsto b^{\epsilon_1}$ are pairwise non-equal modulo the inner automorphisms, and that the ψ_k are also pairwise non-equal modulo the inner automorphisms. It now suffices to prove that no two automorphisms of the form $\phi_1 \psi_j$ and $\phi_2 \psi_k$, with $0 \leq j, k \leq \frac{n}{2}$, are equal modulo the inner automorphisms. However, if this were so then $(a\phi_1 \psi_j, b\phi_1 \psi_j) = (a\psi_j, b\psi_j)\phi_1$ (as ϕ_1 is a Nielsen transformation) would be in the same Nielsen equivalence class as $(a\phi_2 \psi_k, b\phi_2 \psi_k) = (a\psi_k, b\psi_k)\phi_2$, and so $(a\psi_j, b\psi_j)$ and $(a\psi_k, b\psi_k)$ as Nielsen equivalent. Thus, $j = k$. This means that $\psi_k \phi_1 = \psi_k \phi_2 \text{ mod } \text{Inn}(G)$, and so $\phi_1 = \phi_2 \text{ mod } \text{Inn}(G)$. Thus, $\phi_1 = \phi_2$, by Lemma 3.4, as required.

Now, we have that the automorphisms $\alpha, \beta, \delta, \zeta$ and ψ_k for $0 < k < n$, $\gcd(k, n) = 1$, generate $\text{Out}(G)$. The generators α, β, δ and ζ give the group with presentation,

$$\langle \alpha, \beta, \delta, \zeta; \alpha^2, \beta^2, \zeta^2, [\delta, \zeta], \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \alpha\beta\zeta, \delta^n \rangle$$

by Section 3.2 and because δ has order n (as b has order n). Further,

$$\begin{aligned} \delta\psi_i &= \psi_i\delta^i \\ \alpha\psi_i &= \psi_i\alpha \\ \beta\psi_i &= \psi_i\beta \\ \zeta\psi_i &= \psi_i\zeta \\ \psi_i\psi_j &= \psi_{ij \bmod n} \end{aligned}$$

modulo the inner automorphisms and so

$$\begin{aligned} \text{Out}(G) \cong \langle \alpha, \beta, \delta, \zeta, \psi_i; \beta = \psi_{n-1}, \alpha^2, \beta^2, \zeta^2, \delta^n, [\delta, \zeta], \alpha\delta\alpha = \delta^{-1}, \beta\delta\beta = \delta^{-1}, \alpha\beta\zeta, \\ \psi_i^{-1}\delta\psi_i = \delta^i, [\alpha, \psi_i], [\beta, \psi_i], [\zeta, \psi_i], \psi_i\psi_j = \psi_{ij \bmod n} \rangle. \end{aligned}$$

Replacing ζ with $\alpha\beta$ and β with ψ_{-1} , and following the Tietze transformations through we get

$$\begin{aligned} \text{Out}(G) \cong \langle \alpha, \delta, \psi_i; \alpha^2, \delta^n, \alpha\delta\alpha = \delta^{-1}, \\ \psi_i^{-1}\delta\psi_i = \delta^i, [\alpha, \psi_i], \psi_i\psi_j = \psi_{ij \bmod n} \rangle. \end{aligned}$$

Writing $H = \langle \alpha, \delta \rangle$ and $K = \langle \psi_k \ (0 \leq k < n, \gcd(k, n) = 1); \psi_i\psi_j = \psi_{ij \bmod n} \rangle$, clearly $G = HK$, $H \cap K = \langle 1 \rangle$ and $H \triangleleft G$. Thus, $G = H \rtimes K$.

Clearly $H \cong D_n$ while the group K is the automorphisms group of C_n , $\text{Aut}(C_n)$. This completes the proof. \square

5. TWO-GENERATOR ONE-RELATOR GROUPS WITH TORSION

What is written in Sections 3 and 4 classifies $\text{Out}(G)$ where G is a one-relator group with torsion and the relator is not in $F(a, b)'$, the derived subgroup of $F(a, b)$. We now wish to prove the analogue of Lemma 3.4 for general one-relator groups with torsion, so we include those previously excluded. This proves that if R is not primitive then $\text{Out}(G)$ embeds in $\text{Out}(F(a, b))$ and so is residually finite. From there we prove that if $G \cong \langle a, b; [a, b]^n \rangle$ then $\text{Out}(G) = \text{Out}(F(a, b))$, and that otherwise $\text{Out}(G)$ is virtually cyclic.

5.1. $\text{Out}(G)$ is residually finite.

Theorem 5.1. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and R non-primitive. Then $\text{Out}(G)$ injects into $\text{Out}(F(a, b))$ in a canonical way.*

The interpretation of ‘canonical’ is the interpretation found in Section 2. That is, we prove that the map θ in Figure 2 has trivial kernel.

Proof. Note that Lemma 3.4 proves the result for $R \notin F(a, b)'$. Therefore, we can assume without loss of generality that $R \in F(a, b)'$.

It is sufficient to prove that if ϕ is a Nielsen transformation and $\phi \in \text{Inn}(G)$ then $\phi \in \text{Inn}(F(a, b))$. So, let ϕ be some Nielsen transformation of (a, b) with $a\phi := A$ and $b\phi := B$ and such that there exists $W \in F(a, b)$ with $a^W =_G A$ and $b^W =_G B$.

As $a^W = A$ and $b^W = B$ in G it must hold that $a^W = A \bmod G'$ and $b^W = B \bmod G'$. However, $G^{ab} = \langle a, b; [a, b] \rangle \cong \mathbb{Z} \times \mathbb{Z}$, as $R \in F(a, b)'$. Therefore, it must hold that $\sigma_a(A) = 1$ and $\sigma_b(A) = 0$, and that $\sigma_a(B) = 0$ and $\sigma_b(B) = 1$.

Thus, recalling that under the homomorphism $\xi : \text{Aut}(F(a, b)) \rightarrow GL(2, \mathbb{Z})$, ϕ is mapped to the identity matrix. By Proposition 1.3, this means that $\phi \in \text{Inn}(F(a, b))$, as required. \square

Corollary 5.2. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$. Then $\text{Out}(G)$ is residually finite.*

Proof. Noting that $\text{Out}(F(a, b))$ is residually finite, Theorem 5.1 proves that $\text{Out}(G)$ is isomorphic to a subgroup of a residually finite group. Subgroups of residually finite groups are residually finite, and so $\text{Out}(G)$ is residually finite. \square

Corollary 5.3. *If $G = \langle a, b; [a, b]^n \rangle$ with $n > 1$ then $\text{Out}(G) = \text{Out}(F(a, b))$.*

Proof. By Theorem 3.9 (p165) of [12], if $\phi \in \text{Aut}(F(a, b))$ then $[a, b]\phi = [a, b]^W$ for some $W \in F(a, b)$; that is, every automorphism of $F(a, b)$ defines an automorphism of G . Therefore, the subgroup H of $\text{Aut}(F(a, b))$ in Figure 1 is the whole of $\text{Aut}(F(a, b))$ while Theorem 5.1 means that $\text{Out}(G) = H / \text{Inn}(F(a, b))$, which proves the result. \square

5.2. $\text{Out}(G)$ and Equations in Free Groups. In this subsection we apply Theorem 5.1 to solutions to equations in free groups, which can be found in [14], and in doing so prove that $\text{Out}(G)$ is either finite, $\text{Out}(F(a, b))$, or virtually- \mathbb{Z} .

The solutions to equations in free groups are relevant here because finding automorphisms up to conjugacy corresponds to solving the equations $R(x, y) = R^{\pm 1}(a, b)$ in free groups (x and y are the variables). This follows from Theorem N5 (p172) of [12], and the fact that its converse is also true,

Proposition 5.4. *Let G be a group on generators x_ν ($\nu = 1, 2, \dots, n$) with a single defining relator $R(x_\nu)$. If there exists a set of words*

$W_\nu(x_\mu)$ such that the mapping

$$x_\nu \mapsto W_\nu(x_\mu) \ (\nu = 1, 2, \dots, n)$$

is a Nielsen transformation acting on the x_ν which defines an automorphism of G , then $R(W_\nu)$ is freely equal (as a word in the x_ν) to a transform

$$T(x_\nu) \cdot R(x_\nu)^{\pm 1} \cdot T^{-1}(x_\nu)$$

of $R^{\pm 1}$.

The converse of this Proposition is also true: If there exists a set of words $W_\nu(x_\mu)$ such that the mapping

$$x_\nu \mapsto W_\nu(x_\mu) \ (\nu = 1, 2, \dots, n)$$

is a Nielsen transformation acting on the x_ν and $R(W_\nu)$ is freely equal (as a word in the x_ν) to a transform

$$T(x_\nu) \cdot R(x_\nu)^{\pm 1} \cdot T^{-1}(x_\nu)$$

of $R^{\pm 1}$ then the Nielsen transformation defines an automorphism of G .

This is true because the mapping is a homomorphism, and because it is a Nielsen transformation the fact that it is a homomorphism immediately implies that it is an automorphism.

We now call on the work of Touikan [14]. Specifically, we need the following proposition which outlines the forms a rank 2 solution to the equation $w(x, y) = u$ can take, $u \in F(a, b)$. In the proposition, a *solution* is a map $\phi : x \mapsto x', y \mapsto y', x', y' \in F(a, b)$, such that $w(x', y') \equiv u$, or equivalently a pair (x', y') under the same conditions, while a *rank 2 solution* is a solution such that x', y' are not contained in some cyclic subgroup of $F(a, b)$. A *primitive solution* is a solution (x', y') such that (x', y') is a primitive pair of $F(a, b)$. Two equations $w(x, y) = u$ and $w'(x, y) = u'$ are *rationally equivalent* if there is a Nielsen Transformation of (x, y) , φ say, such that $w\varphi = w'$.

If (t, p) is a primitive pair of $F(x, y)$, write

$$\begin{aligned} \bar{\delta}_t : t &\mapsto pt \\ p &\mapsto p \end{aligned}$$

while γ_v denotes the (inner) automorphism of $F(a, b)$ corresponding to conjugation by v , $\gamma_v : a \mapsto v^{-1}av, b \mapsto v^{-1}bv$.

Proposition 5.5. *Suppose that $w(x, y) = u$ has rank 2 solutions and that $w(x, y)$ is neither primitive nor a proper power. Let $\{\phi_i : i \in I\}$ be a finite set of solutions. Then the rank 2 solutions are given by one of the following,*

- (1) *All solutions are of the form $\phi_i \gamma_u^j$, $j \in \mathbb{Z}$.*

- (2) We have $\langle x, y \rangle = \langle H, t; t^{-1}pt = q \rangle$, with $H = \langle p, q \rangle$, $w \in H$, and we can write the elements x, y as words $x = X(p, t)$, $y = Y(p, t)$. All solutions are of the form

$$\bar{\delta}_t^k \phi_i \gamma_u^j,$$

$j, k \in \mathbb{Z}$.

- (3) Up to rational equivalence, $w(x, y) \equiv [x, y]$ and all solutions are of the form

$$\sigma \phi_i$$

where $\sigma \in \langle \bar{\delta}_x, \bar{\delta}_y, \gamma_w \rangle$.

Applying the fact that the map $\phi : x \mapsto a, y \mapsto b$ is a Δ -minimal solution to the equation $R(x, y) \equiv R(a, b)$ (see [14] for the definition of Δ -minimal. If ϕ is Δ -minimal then $\phi \in \{\phi_i : i \in I\}$ the finite set of solutions) we can abuse notation to equate x with a and y with b , so we can write $\bar{\tau} \circ \phi = \bar{\tau}$, $\phi \circ \tau = \tau$, and $w\phi = w$ for $w \in F(a, b)$, and working mod $\text{Inn}(G)$ we see that if $G = \langle a, b; R^n \rangle$ then one of only three things must happen,

- (1) There are only finitely many solutions to $R(x, y) \equiv R^{\pm 1}(a, b)$,
- (2) $G \cong \langle a, b; S^n \rangle$, $S \in \langle a^{-1}ba, b \rangle$,
- (3) $G \cong \langle a, b; S^n \rangle$ with $S = [a, b]$.

We now use Touikan's solutions to prove the following result about the structure of $\text{Out}(G)$,

Theorem 5.6. *If G is a two-generator one-relator group with torsion then either,*

- $\text{Out}(G)$ is virtually cyclic,
- $\text{Out}(G) = \text{Out}(F(a, b))$ and $G \cong \langle a, b; [a, b]^n \rangle$.

Proof. Firstly, note that if $\text{Out}(G)$ is finite then it is trivially virtually cyclic. So we restrict ourselves to the case where $\text{Out}(G)$ is infinite; to the second two cases of Touikan's solution to $R(x, y) \equiv R(a, b)$. If the third case of Touikan's solution holds, then $\text{Out}(G) = \text{Out}(F(a, b))$ by Corollary 5.3.

We prove that if the second case of Touikan's solution holds then $\text{Out}(G)$ is virtually- \mathbb{Z} . We know that $\text{Out}(G)$ is infinite because we can re-write R in terms of p and t , and then $\bar{\delta}_t \in \text{Aut}(G)$. By Theorem 5.1 no power of $\bar{\delta}_t$ is inner, thus $\bar{\delta}_t$ has infinite order in $\text{Out}(G)$. We essentially prove that the subgroup $\langle \bar{\delta}_t \text{Inn}(G) \rangle$ has finite index in $\text{Out}(G)$.

Assume the second case of Touikan's solution holds, and define \mathcal{S}_p to be the set of automorphisms of G which (freely) fix R or send it to R^{-1} ; one can view this set as the set of primitive solutions

to $R(x, y) \equiv R(a, b)$ unioned with the set of primitive solutions to $R(x, y) \equiv R(a, b)^{-1}$. Clearly, \mathcal{S}_p is closed under products and inverses, and contains the identity automorphism, so $\mathcal{S}_p \leq \text{Aut}(G)$, and noting that $\mathcal{S}_p \text{Inn}(G) = \text{Aut}(G)$, this means that

$$\text{Out}(G) \cong \frac{\mathcal{S}_p}{\mathcal{S}_p \cap \text{Inn}(G)}$$

and so we prove that $\mathcal{S}_p / \mathcal{S}_p \cap \text{Inn}(G)$ is virtually cyclic.

Now, \mathcal{S}_p contains as a normal subgroup $\text{Stab}_p(R) := \mathcal{S}_p \cap \text{Stab}(R)$, the *primitive stabiliser* of R . Proposition 2.21 of [14] tells us that if $\phi_0, \phi_1 \in \mathcal{S}_p$ and ϕ_0, ϕ_1 have the same *terminal pair* and the same *terminal word* then there exists some element $\beta \in \text{Stab}_p(R)$ such that $\phi_0\beta = \phi_1$, while Proposition 2.19 of [14] gives us that there are only finitely many possible terminal pairs and terminal words. That is, $\text{Stab}_p(R)$ is of finite index in \mathcal{S}_p . Thus, we prove that $\text{Stab}_p(R) / \mathcal{S}_p \cap \text{Inn}(G)$ is virtually cyclic.

Clearly $N = \langle \gamma_R \rangle \leq \text{Inn}(G)$, so $N \cap \text{Inn}(G) = N$, while $N \trianglelefteq \text{Stab}_p(R)$, because if $R\phi \equiv R^\epsilon$ then $\gamma_R\phi = \phi\gamma_{R^\epsilon}$. Therefore,

$$\text{Stab}_p(R) \twoheadrightarrow \frac{\text{Stab}_p(R)}{N} \twoheadrightarrow \frac{\text{Stab}_p(R)}{\mathcal{S}_p \cap \text{Inn}(G)}$$

and as a quotient of a virtually cyclic group is virtually cyclic, to prove that $\text{Out}(G)$ is virtually cyclic it suffices to prove that $\text{Stab}_p(R)/N$ is virtually cyclic.

Looking at [14], Corollary 2.12 and Section 2.4.1 combine to give us that $\Delta = \langle \gamma_R, \delta_t \rangle$ is of finite index in $\text{Stab}_p(R)$. Thus, as Δ/N is virtually cyclic so is $\text{Stab}_p(R)/N$, as required. \square

6. WHAT DOES $\text{Aut}(G)$ LOOK LIKE, FOR $R \notin F(a, b)'$?

In Sections 3 and 4 we found $\text{Out}(G)$ for $G = \langle a, b; R^n \rangle$, $R \notin F(a, b)'$, $n > 1$. However, in these sections we could have computed $\text{Aut}(G)$ instead. We do this now.

Getting a presentation for $\text{Aut}(G)$ via $\text{Out}(G)$ is easy, because G is centerless by [2], and by Lemma 3.2 every automorphism is of the form $\phi_i\gamma_w$, where $\phi_i \in \{\alpha_i, \beta_i, \zeta_i, \delta_i, e\}$ for some $i \in \mathbb{Z}$ and γ_w denotes conjugation by w for some word $w \in F(a, b)$, because $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$. Further, if $\phi_i\gamma_w = \phi'_j\gamma_v$ then $\phi_i = \phi'_j \bmod \text{Inn}(G)$, and so ϕ_i and ϕ'_j are the same automorphisms by Lemma 3.4.

It is therefore easy to write down $\text{Aut}(G)$, as it is (essentially) dependent only on which of $\alpha_i, \beta_i, \zeta_i$ and δ_i are in it. To do this,

- The inner automorphisms are isomorphic to G in the canonical way (as G has trivial centre), so we immediately have the

relation

$$\gamma_{R^n} = 1.$$

- We have that

$$\gamma_w^\psi = \gamma_{w\psi},$$

that is, the automorphisms act on the inner automorphisms in the natural way. This is because if $d\psi^{-1} = D$ for $d \in \{a, b\}$ then

$$\begin{aligned} d\psi^{-1}\gamma_w\psi &= D\gamma_w\psi \\ &= (w^{-1}Dw)\psi \\ &= (w^{-1}\psi)d(w\psi) \\ &= d\gamma_{w\psi} \end{aligned}$$

as required. Therefore, we have the relations

$$\begin{aligned} \gamma_a^{\phi_i} &= \gamma_{a\phi_i} \\ \gamma_b^{\phi_i} &= \gamma_{b\phi_i} \end{aligned}$$

for all $\phi_i \in \text{Aut}(G) \cap \{\alpha_i, \beta_i, \zeta_i, \delta_i\}$.

- We have to ascertain how the ϕ_i multiply together, but we have done much of this in Section 3.2: we know the relations modulo the inner automorphisms. This means we just need to add in the inner automorphism a relator in $\text{Out}(G)$ is equal to. For example, $\alpha_i^2 = \gamma_{b^i}$ while $\beta_i^2 = 1$, and $\delta_0^{\alpha_0}\delta_0 = \gamma_b$ while $\delta_0^{\beta_0}\delta_0 = 1$.
- These are all the relations, as any other non-trivial relation would be of the form either

$$U(\gamma_a, \gamma_b) = 1$$

where $U(a, b) \neq_G 1$ or

$$V(\gamma_a, \gamma_b) = \phi_i.$$

However, the former cannot happen as $G \cong \text{Inn}(G)$ under the isomorphism $a \mapsto \gamma_a$, $b \mapsto \gamma_b$, because G is centerless, while the latter cannot happen as it corresponds to a relator in $\text{Out}(G)$ and we have captured all of these.

Writing α for α_0 , β for β_0 and ζ for ζ_0 , and writing w for γ_w (so w represents the automorphism corresponding to conjugation by w) we have the following cases. Firstly, if R is non-primitive then,

- (1) $\text{Aut}(G) = \langle \alpha_i, \text{Inn}(G) \rangle$ and
 $\text{Aut}(G) = \langle \alpha_i, a, b; R^n, \alpha_i^2 = b^i, a^{\alpha_i} = a^{-1}b^i, b^{\alpha_i} = b \rangle$
- (2) $\text{Aut}(G) = \langle \beta_i, \text{Inn}(G) \rangle$ and
 $\text{Aut}(G) = \langle \beta_i, a, b; R^n, a^{\beta_i} = ab^i, b^{\beta_i} = b^{-1}, \beta_i^2 = 1 \rangle$
 $\cong G \rtimes C_2$
- (3) $\text{Aut}(G) = \langle \zeta, \text{Inn}(G) \rangle$ and
 $\text{Aut}(G) = \langle \zeta, a, b; R^n, a^\zeta = a^{-1}, b^\zeta = b^{-1}, \zeta^2 = 1 \rangle$
 $\cong G \rtimes C_2$
- (4) $\text{Aut}(G) = \langle \alpha_i, \beta_{-i}, \zeta, \text{Inn}(G) \rangle$ and
 $\text{Aut}(G) = \langle \alpha_i, \beta_{-i}, \zeta, a, b; R^n,$
 $a^{\alpha_i} = a^{-1}b^i, b^{\alpha_i} = b, \alpha_i^2 = b^i$
 $a^{\beta_{-i}} = ab^{-i}, b^{\beta_{-i}} = b^{-1}, \beta_{-i}^2 = 1,$
 $a^\zeta = a^{-1}, b^\zeta = b^{-1}, \zeta^2 = 1,$
 $\alpha_i \beta_{-i} = \zeta b^{-i}, \alpha_i \zeta = \zeta \alpha_i b^{-i},$
 $\beta_{-i} \alpha_i = \zeta, \zeta \beta_{-i} = \alpha_i b^{-i} \rangle$
- (5) $\text{Aut}(G) = \langle \alpha, \delta, \text{Inn}(G) \rangle$ and
 $\text{Aut}(G) = \langle \alpha, \delta, a, b; R^n,$
 $a^\alpha = a^{-1}, b^\alpha = b, \alpha^2 = 1, a^\delta = ab, b^\delta = b, \delta^\alpha = \delta^{-1}b \rangle$
- (6) $\text{Aut}(G) = \langle \beta, \delta, \text{Inn}(G) \rangle$ and
 $\text{Aut}(G) = \langle \beta, \delta, a, b; R^n,$
 $a^\beta = a, b^\beta = b^{-1}, \beta^2 = 1, a^\delta = ab, b^\delta = b, \delta^\beta = \delta^{-1} \rangle$
 $\cong G \rtimes D_\infty$

$$\begin{aligned}
(7) \quad & \text{Aut}(G) = \langle \zeta, \delta, \text{Inn}(G) \rangle \text{ and} \\
& \text{Aut}(G) = \langle \zeta, \delta, a, b; R^n, \\
& \quad a^\zeta = a^{-1}, b^\zeta = b^{-1}, \zeta^2 = 1, a^\delta = ab, b^\delta = b, \delta^\zeta = \delta b^{-1} \rangle \\
(8) \quad & \text{Aut}(G) = \langle \alpha, \beta, \zeta, \delta, \text{Inn}(G) \rangle \text{ but one can write } \zeta = \alpha\beta \\
& \quad \text{and so we end up with} \\
& \text{Aut}(G) = \langle \alpha, \beta, \delta, a, b; R^n, \\
& \quad a^\alpha = a^{-1}, b^\alpha = b, \alpha^2 = 1, \\
& \quad a^\beta = a, b^\beta = b^{-1}, \beta^2 = 1, \\
& \quad a^\delta = ab, b^\delta = b, \\
& \quad [\alpha, \beta] = 1, \\
& \quad \delta^\alpha = \delta^{-1}b, \delta^\beta = \delta^{-1} \rangle
\end{aligned}$$

While, if R is primitive we have,

$$\begin{aligned}
(9) \quad & \text{Aut}(G) = \langle \alpha, \beta, \zeta, \delta, \text{Inn}(G) \rangle, \text{ but one can write } \zeta = \alpha\beta \\
& \quad \text{and so we end up with} \\
& \text{Aut}(G) = \langle \alpha, \beta, \delta, a, b; b^n, \\
& \quad a^\alpha = a^{-1}, b^\alpha = b, \alpha^2 = 1, \\
& \quad a^\beta = a, b^\beta = b^{-1}, \beta^2 = 1, \\
& \quad a^\delta = ab, b^\delta = b, \delta^n = 1, \\
& \quad [\alpha, \beta] = 1, \\
& \quad \delta^\alpha = \delta^{-1}b, \delta^\beta = \delta^{-1} \rangle
\end{aligned}$$

7. GENERALISING TO OTHER GROUPS

The ideas used in this paper generalise to certain other groups, as can be seen by Theorem 2.3. We give some two-generated examples. For one of these examples we can apply almost identical arguments as we applied to groups of the form $\langle a, b; R^n \rangle$, $n > 1$ and $R \notin F(a, b)'$ to look at $\text{Out}(G)$, while the other class is investigated using Lemma 3.2.

Now, the question of when a group possesses only finitely many Nielsen Equivalence Classes, and so satisfies Condition (1) of Theorem 2.3, has been much studied. A summary of some of the results in this area can be found in [7]. We therefore wish to concentrate on determining when a group satisfies Condition (2) of Theorem 2.3.

Lemma 7.1. *If $G = \langle a, b; \mathbf{r} \rangle$ with $\mathbf{r} \subseteq F(a, b)'$ then G satisfies Condition (3).*

Proof. The proof is identical to the proof of Theorem 5.1, as $\mathbf{r} \subseteq F(a, b)'$ so we have $G^{ab} \cong \mathbb{Z} \times \mathbb{Z}$. \square

Lemma 7.2. *If $G = \langle a, b; \mathbf{r} \rangle$ such that $\sigma_a(R) = 0$ for all $R \in \mathbf{r}$ but $\mathbf{r} \not\subseteq F(a, b)'$ then $L/\text{Inn}(G)$ is virtually cyclic and G satisfies Condition (2).*

Proof. Lemma 3.2 holds in this case, as does Figure 4 (page 12).

Now, looking at Figure 4, $K/\text{Inn}(F(a, b)) \cong D_\infty \times C_2$ by Section 3.2, so $K/\text{Inn}(F(a, b))$ is virtually cyclic. As subgroups of virtually cyclic groups are virtually cyclic, $H/\text{Inn}(F(a, b))$ is also virtually cyclic. As homomorphic images of virtually cyclic groups are virtually cyclic, $L/\text{Inn}(G)$ is virtually cyclic, as required. \square

These two Lemmata, along with Theorem 2.3, allow us to prove the following theorem,

Theorem 7.3. *If $G = \langle a, b; \mathbf{r} \rangle$ is a two-generator group with infinite abelianisation and with only finitely many Nielsen Equivalence Classes in the T -system of (a, b) then $\text{Out}(G)$ is residually finite. If $\mathbf{r} \not\subseteq F(a, b)'$ then $\text{Out}(G)$ is virtually cyclic.*

Proof. Noting that if $G = \langle a, b; \mathbf{r} \rangle$ is a two-generator group then $G^{ab} \cong \mathbb{Z} \times \mathbb{Z}$ if and only if $\mathbf{r} \subseteq F(a, b)'$, it is therefore sufficient to prove that every two-generator group with abelianisation $\mathbb{Z} \times C_n$ for some $n \in \mathbb{N} \cup \{0\}$ has a presentation of the form $\langle a, b; \mathbf{s} \rangle$ with $\sigma_a(S) = 0$ for all $S \in \mathbf{s}$, and that this presentation can be got to via a Nielsen transformation on (a, b) .

To prove this, if $G^{ab} \cong \langle x, y; y^n, [x, y] \rangle$ and $\pi : G \rightarrow \langle x, y; y^n, [x, y] \rangle$, it is sufficient to find a Nielsen transformation $\phi : (a, b) \mapsto (A, B)$ such that $A\pi = x$. This is sufficient. To see this sufficiency, note that $B\pi$ is then forced to have finite order. We can re-write the presentation for G as $\langle A, B; \mathbf{s} \rangle$ for some set of relators \mathbf{s} , and if there exists some $S \in \mathbf{s}$ such that $\sigma_A(S) = i \neq 0$ then we have that $(A\pi)^i = (B\pi)^j$, where $\sigma_B(S) = j \in \mathbb{Z}$. However, this means that $A\pi$ has finite order, a contradiction.

So, we wish to prove that there exists some Nielsen transformation $\phi : (a, b) \mapsto (A, B)$ such that $A\pi = x$. Now, $G^{ab} = \langle a, b; (a^p b^q)^n, [a, b] \rangle \cong \langle x, y; y^n, [x, y] \rangle$. As we have mentioned already, at the start of Section 3, there exists an algorithm which takes a word $R(a, b)$ and re-writes it such that $\sigma_a(R) = 0$ (see, for example, [10]). This algorithm uses Nielsen transformations. Therefore, taking $R(a, b) = a^p b^q$, we apply the Nielsen transformations which re-write R such that $\sigma_a(R) = 0$ to

the presentation $\langle a, b; \mathbf{r} \rangle$. When we then computer the abelinisation of this new presentation for G we have that the image of b in G^{ab} has finite order, as required. \square

This leaves us with the following question,

Question 1. *Does there exist a group $G = \langle a, b; \mathbf{r} \rangle$ where G^{ab} is finite and $L/\text{Inn}(G)$ is not residually finite?*

We now turn to small cancellation groups (see, for example, [11]). By [5], if G admits a 2-generator $C'(1/14)$ or a 2-generator $C'(1/10) - T(4)$ presentation then G has only finitely many Nielsen Equivalence Classes, and so satisfies Condition (1). We therefore wish to see when a $C'(\lambda)$ group satisfies Condition (2).

Example 7.4. *If $G = \langle a, b; \mathbf{r} \rangle$, G has infinite abelinisation, and \mathbf{r}^* satisfies $C'(1/14)$ or $C'(1/10) - T(4)$, then $\text{Out}(G)$ is residually finite.*

Proof. As \mathbf{r}^* satisfies $C'(1/14)$ or $C'(1/10) - T(4)$, G satisfies Condition (1) by [5], while we have that G satisfies Condition (2) by Theorem 7.3, as required. \square

We look at a specific class of small-cancellation groups: groups with presentation $\langle a, b; \mathbf{r} \rangle$ where \mathbf{r}^* satisfies $C'(1/24)$, all elements of \mathbf{r} are proper powers, and $\mathbf{r} \subseteq \langle a^{-1}ba, b \rangle \cup \langle aba^{-1}, b \rangle$. Such a group has residually finite outer automorphism group by the above, and if $\mathbf{r} \not\subseteq F(a, b)'$ then we can treat such a group very similarly to two-generator one-relator groups with torsion where the relator is not contained in $F(a, b)'$.

Firstly, however, we need a lemma, the proof of which needs the following Proposition (which is Lemma 2.17 (p48) of [15]),

Proposition 7.5. *Let $\langle X, a, b; \mathbf{r} \rangle$ satisfying $C'(1/6)$. Let $c \in \{a, b\}$. Then the equation $a^\alpha W c^\beta W^{-1} =_G 1$ holds only if either $a^\alpha = c^\beta = 1$ or $W =_G a^\gamma c^\delta$ for some γ, δ .*

Lemma 7.6. *Let $G = \langle a, b; \mathbf{r} \rangle$ be non-cyclic with \mathbf{r} satisfying $C'(1/6)$, $\mathbf{r} \not\subseteq F(a, b)'$ and such that for all $R \in \mathbf{r}$, $\sigma_a(R) = 0$. Then,*

- *If b has infinite order in G then the automorphisms $\phi \in \text{Aut}(G)$ such that $(a\phi, b\phi)$ lie in the set $S = \{(a^{\epsilon_0} b^k, b^{\epsilon_1}) : k \in \mathbb{Z}\}$ form a transversal for $L/\text{Inn}(G)$,*
- *If b has order $n < \infty$ in G then the automorphisms $\phi \in \text{Aut}(G)$ such that $(a\phi, b\phi)$ lie in the set $S = \{(a^{\epsilon_0} b^k, b^{\epsilon_1}) : 0 \leq k < n\}$ form a transversal for $L/\text{Inn}(G)$.*

Proof. Note that if $U =_G 1$ then $\sigma_a(U) = 0$. Otherwise, letting $\pi : a \mapsto x, b \mapsto y$ be the abelianisation map, one has that $U\pi = x^i y^j, i \neq 0$ with $x^i y^j = 1$. Now, G^{ab} has presentation $\langle x, y; y^m, [x, y] \rangle, m \in \mathbb{N} \cup \{0\}$, so if $x^i y^j = 1$ we must have that $i = 0$, a contradiction.

By Lemma 3.2 it is sufficient in each case (b having infinite or finite order) to prove that the elements of our prospective transversal are non-equal mod $\text{Inn}(G)$. This is equivalent to proving that for all ϵ_0, ϵ_1 and $k \in \mathbb{Z}$ there does not exist a $w \in F(a, b)$ such that $a^w =_G a^{\epsilon_0} b^k, b^w =_G b^{\epsilon_1}$, and $(a, b) \not\equiv (a^{\epsilon_0} b^k, b^{\epsilon_1})$. This equivalence is because if $\phi_1 = \phi_2 \text{ mod } \text{Inn}(G)$ and $\phi_1 \neq \phi_2$ as automorphisms then $\phi_1 \phi_2^{-1} \in \text{Inn}(G)$ but is non-trivial. Then ϕ can be taken to be

$$\begin{aligned}\phi : a &\mapsto a^{\epsilon_0 \epsilon'_0} b^{\epsilon'_1 (j - \epsilon_0 \epsilon'_0 i)} \\ b &\mapsto b^{\epsilon_1 \epsilon'_1}\end{aligned}$$

which is equal to $\phi_1 \phi_2^{-1}$ modulo the inner automorphisms, and is non-trivial unless $\epsilon_0 = \epsilon'_0, \epsilon_1 = \epsilon'_1$, and $i = j$ if b has infinite order, or $i = j \text{ mod } n$ if b has order n .

Now, we investigate our two cases,

- (1) If b has infinite order in G then we prove that the automorphisms $\phi \in \text{Aut}(G)$ such that $(a\phi, b\phi)$ lie in the set $S = \{(a^{\epsilon_0} b^k, b^{\epsilon_1}) : k \in \mathbb{Z}\}$ form a transversal for $L/\text{Inn}(G)$. Call the set of all such automorphisms T .

Assume that there exists a pair from $S, (a^{\epsilon_0} b^k, b^{\epsilon_1})$ say, such that the pair is not freely equal to the pair (a, b) but that these pairs are conjugate in $G, (a^w, b^w) =_G (a^{\epsilon_0} b^k, b^{\epsilon_1})$.

So, if $w =_G 1$ then we have that $b =_G b^{\epsilon_1}$, so $\epsilon_1 = 1$ as b is of infinite order. Further, as $a = a^{\epsilon_0} b^k$ we have that $a^{-1 + \epsilon_0} b^k =_G 1$. Thus, $\epsilon_0 = 1$ (as if $U =_G 1$ then $\sigma_a(U) = 0$). Therefore, $b^k = 1$, so $k = 0$ as b is of infinite order. However we then have that $a \equiv a^{\epsilon_0} b^k$ and $b \equiv b^{\epsilon_1}$ as $\epsilon_0 = 1, \epsilon_1 = 1$ and $k = 0$.

If $w \neq_G 1$ then as $b^w = b^{\epsilon_1}$ we can apply Proposition 7.5 to get that $w =_G b^i$ for some $i \in \mathbb{Z}$. Therefore, $b^{-i} a b^i =_G a^{\epsilon_0} b^k$, so $b^{-i} a b^i b^{-k} a^{-\epsilon_0} =_G 1$. Therefore, $\epsilon_0 = 1$, so b^i is conjugate to b^{i-k} and again using Proposition 7.5 we have that this conjugating element is a power of b . Thus, $b^s =_G a$ for some $s \in \mathbb{Z}$, or $i = k$. Clearly $b^s \neq_G a$ as $\sigma_a(b^s a^{-1}) \neq 0$, so $i = k$. We therefore have that $b^{-i} =_G 1$. As b has infinite order in G , this contradicts the assumption that $b^i = w \neq_G 1$.

- (2) If b has order $n < \infty$ in G then we prove that the automorphisms $\phi \in \text{Aut}(G)$ such that $(a\phi, b\phi)$ lie in the set $S =$

$\{(a^{\epsilon_0}b^k, b^m) : 0 \leq k < n, m \in \{1, n-1\}\}$ form a transversal for $L/\text{Inn}(G)$. Call the set of all such automorphisms T .

Assume that there exists a pair from S , $(a^{\epsilon_0}b^k, b^m)$ say, such that the pair is not freely equal to the pair (a, b) but that these pairs are conjugate in G , $(a^w, b^w) =_G (a^{\epsilon_0}b^k, b^m)$.

So, if $w =_G 1$ then we have that $b =_G b^{\epsilon_1}$, so either we immediately have $n = 1$, or $\epsilon_1 = 1$, or we have that $b^2 = 1$. The case of $n = 1$ cannot happen, as then G is cyclic, while if $b^2 = 1$ then $1 = -1 \pmod n$ and so we can assume $\epsilon_1 = 1$. Thus, $\epsilon_1 = 1$. Further, as $a = a^{\epsilon_0}b^k$ we have that $a^{-1+\epsilon_0}b^k =_G 1$. Thus, $\epsilon_0 = 1$ (as if $U =_G 1$ then $\sigma_a(U) = 0$). Therefore, $b^k = 1$, so $n \mid k$ as b is of order n . As $0 \leq k < n$, $k = 0$. However we then have that $a \equiv a^{\epsilon_0}b^k$ and $b \equiv b^m$, as $\epsilon_0 = 1$, $\epsilon_1 = 1$ and $k = 0$.

If $w \neq_G 1$ then as $w^{-1}bw = b^{\epsilon_1}$ we can apply Proposition 7.5 to get that $w =_G b^i$ for some $i \in \mathbb{Z}$. Therefore, $b^{-i}ab^i = a^{\epsilon_0}b^k$, so $b^{-i}ab^ib^{-k}a^{-\epsilon_0} =_G 1$. Therefore, $\epsilon_0 = 1$, so b^i is conjugate to b^{i-k} and again using Proposition 7.5 we have that this conjugating element is a power of b . Thus, $b^s =_G a$ for some $0 \leq s < n$, or $b^{i-k} =_G 1$. Clearly $b^s \neq_G a$ as $\sigma_a(b^sa^{-1}) \neq 0$, so $b^{i-k} =_G 1$. We therefore have that $ab^{i-k}a^{-1} =_G b^i$ so $b^i =_G 1$. This contradicts the assumption that $b^i = w \neq_G 1$.

□

Theorem 7.7. *Let G be given by a presentation $\langle a, b; \mathbf{r} \rangle$ where \mathbf{r}^* satisfies $C'(1/24)$, all elements of \mathbf{r} are proper powers, and $\mathbf{r} \subseteq \langle a^{-1}ba, b \rangle \cup \langle aba^{-1}, b \rangle$. Then $\text{Out}(G)$ is residually finite. If $\mathbf{r} \not\subseteq F(a, b)'$ and b has infinite order then $\text{Out}(G)$ is isomorphic to one of the following groups,*

- \mathbb{Z}
- $\mathbb{Z} \times C_2$
- D_∞
- $D_\infty \times C_2$

while if the order of b is $n < \infty$ then $\text{Out}(G) \leq \text{Out}(\langle a, b; b^n \rangle)$ and n divides $|\text{Out}(G)|$.

Proof. If $\mathbf{r} \subseteq F(a, b)'$ then we proved in Example 7.4 that $\text{Out}(G)$ is residually finite. So, we can assume $\mathbf{r} \not\subseteq F(a, b)'$.

Theorem 2.14 (p41) of [15] proves that if the order of b is $n < \infty$ then every generating pair is Nielsen equivalent to a pair of the form (a, b^μ) , $\gcd(\mu, n) = 1$, while if b has infinite order then G has only one Nielsen Equivalence Class.

We split the proof of our theorem into two cases: b has infinite order, and b has finite order.

- (1) Assume that b has infinite order. Then G has only one Nielsen Equivalence Class, so by Lemma 7.6 we have that the set T of automorphisms $\phi \in \text{Aut}(G)$ such that $(a\phi, b\phi) \in \{(a^{\epsilon_0}b^k, b^{\epsilon_1}) : k \in \mathbb{Z}\}$ forms a transversal for $\text{Out}(G)$. As $\mathbf{r} \subseteq \langle a^{-1}ba, b \rangle \cup \langle aba^{-1}, b \rangle$ we have that $\delta : a \mapsto ab, b \mapsto b$ is in $\text{Aut}(G)$, and so it is in T . Thus, we can apply the working from Section 3.2 to get that $\text{Out}(G)$ is one of,
- \mathbb{Z}
 - $\mathbb{Z} \times C_2$
 - D_∞
 - $D_\infty \times C_2$
- as required.
- (2) Assume that b has order $n \in \mathbb{N}$. Then G has $\varphi(n)$ Nielsen Equivalence Classes, but they are not necessarily in the same T -System of G . As with the case where b has infinite order, we note that Lemma 7.6 gives us that the set T of automorphisms $\phi \in \text{Aut}(G)$ such that $(a\phi, b\phi) \in \{(a^{\epsilon_0}b^k, b^{\epsilon_1}) : 0 \leq k < n\}$ forms a transversal for $L/\text{Inn}(G)$ and the map $\delta : a \mapsto ab, b \mapsto b$ is in T . Similarly to above, this allows us to apply the working from Section 4, and so $\text{Out}(G) \leq \text{Out}(\langle a, b; b^n \rangle)$. Note that δ has order n in $L/\text{Inn}(G)$, and so has order n in $\text{Out}(G)$. Therefore, n divides $|\text{Out}(G)|$.

□

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